ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION.

The eighth annual meeting of the Rocky Mountain section was held at the Steel Works Y. M. C. A., Pueblo, Colorado, on March 28 and 29. There were sixteen present, including the following eight members of the Association: I. M. DeLong, Philip Fitch, G. W. Gorrell, G. H. Light, S. L. Macdonald, J. Q. McNatt, H. E. Russell, C. H. Sisam. The section voted to hold the next meeting at the University of Wyoming. The following officers were elected: J. C. FITTERER, chairman; S. L. MACDONALD, vice-chairman; PHILIP FITCH, secretary; G. H. LIGHT, treasurer.

On Friday evening Mr. F. E. Parks, Manager of the Steel Works, delivered an address of welcome. He pointed out the advantages to all concerned of having the members of the section as guests of the company. Professor S. L. Macdonald responded to this address in a fitting manner, assuring Mr. Parks that the company's problems were also those of the section, and expressed the members' appreciation of the company's generous hospitality. An organ recital, followed by an address by Mr. D. K. Dunton, concluded the evening session. On Saturday morning the members visited the Steel Plant, Mr. Louis Deesz of the company officiating as guide.

The following nine papers were read:

(1) "Report of the Cincinnati meeting" by Professor H. E. RUSSELL.

(2) "The undergraduate mathematics club" by Professor S. L. MACDONALD.

(3) "To compute the radius of the circle inscribed in the area bounded by the arcs of three mutually tangent circles" by Mr. J. Q. MCNATT.

(4) "Misleading definitions of 'f' in the elementary theory for finding the envelope of $f(x, y, c) = 0$" by Professor I. M. DELONG.

(5) "A problem in probability" by Professor G. W. GORRELL.

(6) "Pedal curves and related envelopes" by Mr. PHILIP FITCH.

(7) "On curves whose first polars have a rectilinear component" by Professor C. H. SISAM.

(8) "Times of rising and setting of the planets" by Dean H. A. HOWE.

(9) "Magic squares of the first nine orders" by Professor F. H. LOUD.

In the absence of the authors, the papers by Dean Howe and Professor Loud were read respectively by Professors Russell and Sisam.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. In his report, Professor Russell commented on the attendance and interest of the Cincinnati meeting and dwelt briefly on the salient features of some of the more interesting papers.

2. Professor Macdonald's paper dealt with the advisability of having undergraduate mathematics clubs and showed how interest in mathematics was stimulated by such organizations. The meeting of pupils with a common interest often reveals qualities in them that would otherwise be dormant.

3. The paper by Mr. McNatt demonstrated an interesting method of computing the radius of the circle inscribed in the area bounded by three mutually tangent circles, in terms of the radii of these circles.

4. Professor DeLong pointed out that there were definitions of "f," as applied to the elementary theory for finding the envelope of $f(x, y, c) = 0$, that were misleading and remarked that some authors of works on calculus had made no attempt to clarify the subject.

5. Professor Gorrell compared methods of attacking problems in probability and discussed the advantages of having more than one viewpoint of a problem.

6. In his paper, Mr. Fitch demonstrated a short method for finding the equation of a pedal curve and proved the following properties: (*a*) The pedal of a given curve with respect to a fixed point is the envelope of a family of circles described on the radii vectores from the fixed point to the given curve as diameters. (*b*) The fixed point is a conjugate point of this envelope. (*c*) The caustic of a given curve with respect to a fixed point is a translation of the evolute of its pedal for that point.

7. In this paper, Professor Sisam determined the equations of the non-composite algebraic plane curves which have the property that every line through a fixed point is a component of a first polar with respect to the curve.

8. Dean Howe's paper dealt with the computation of the approximate times of rising and setting of the planets. The object of the method set forth is to render it possible for a student in elementary descriptive astronomy, by using data easily taken from the American Ephemeris, to obtain the time of rising or setting of any planet on any day of the year, with an error not exceeding three minutes. No logarithmic or trigonometric work is needed. The place for which the computation is to be made is supposed to be in the northern hemisphere, and to have a latitude no greater than 60°. Denver was chosen to illustrate the process.

From the Greenwich time when the planet crosses the Greenwich meridian on the given date, the Denver time when it crosses the Denver meridian is obtained by a simple interpolation. Then the problem is quickly finished by using the tables for sunrise and sunset at the end of the Ephemeris, making allowance for the fact that these tables are for the upper limb of the sun, instead of the center. In those infrequent cases where a planet's distance from the celestial equator exceeds twenty-three and a half degrees, an extrapolation is necessary.

9. In this paper, Professor Loud derived several new and interesting ways for forming magic squares.

PHILIP FITCH, *Secretary.*

NINTH ANNUAL MEETING OF THE OHIO SECTION.

The ninth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on April 4, 1924, in connection with the meetings of the Ohio College Association and allied societies. Chairman W. E. Anderson presided, being relieved by V. B. Caris for an interval.

Sixty persons were registered, the following thirty-three being members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, Grace M. Bareis, C. T. Bumer, W. D. Cairns, V. B. Caris, E. H. Clarke, H. L. Coar, O. L. Dustheimer, B. C. Glover, G. P. Harmount, H. W. Kuhn, H. B. Lemon, Anna D. Lewis, E. S. Manson, C. N. Moore, C. C. Morris, M. A. Nordgaard, S. E. Rasor, P. L. Rea, C. N. Reynolds, Hortense Rickard, W. G. Simon, S. A. Singer, K. D. Swartzel, M. O. Tripp, R. B. Wildermuth, F. B. Wiley, J. H. Weaver, C. O. Williamson, B. F. Yanney.

At the business session the secretary reported a membership of eighty-five and nine institutional members as against eighty-five and ten, respectively, last year. Officers elected for this year are: Chairman, Professor HARRIS HANCOCK, University of Cincinnati; Secretary-Treasurer, Professor G. N. ARMSTRONG, Ohio Wesleyan University; Third member of the executive committee, Professor R. B. WILDERMUTH, Capital University. Professor R. B. Allen was reëlected to serve on the program committee, his term of service being three years. A collection of $12.00 was taken at the meeting for the immediate uses of the Section. Professor R. B. Wildermuth was named to fill the place of Professor A. D. Pitcher, deceased since the last meeting, upon the committee to issue the usual letters to the high schools of the state on "Why Elect Mathematics?"

The Section dinner, with about fifty present, served in one of the dining rooms of Campbell Hall, was very successful. The evening session was held in the same room and was devoted to the last two numbers on the program.

The following ten papers were presented at the two sessions:

(1) Chairman's Address: "Some methods of creating and maintaining interest in mathematics" by Professor W. E. ANDERSON, Miami University.

(2) "Business statistics" by Mr. EDMOND E. LINCOLN, Ph.D., Chief Statistician, The Western Electric Company, New York (by invitation).

(3) "Related variables with coefficient of correlation equal to zero" by Professor C. N. MOORE, University of Cincinnati.

Business and Intermission.

(4) "Some problems in the teaching of the mathematics of investment" by Professor C. N. REYNOLDS, Jr., University of West Virginia.

(5) "Some notes on mathematics in the investment houses" by Professor G. N. ARMSTRONG, Ohio Wesleyan University.

(6) "Some dual theorems of the quadratic function" by Professor C. L. ARNOLD, Ohio State University.

(7) "An experiment in sectionizing freshman mathematics" by Professor HAZEL SCHOONMAKER, The Western College for Women.

(8) "The use of prognostic tests in sectionizing freshmen in mathematics" by Professor M. A. NORDGAARD, Antioch College.

(9) Reports of the Yanney Committee on the Mathematical Situation in Ohio, continued from 1923. Professor C. N. MOORE, University of Cincinnati, General Chairman.

(10) Discussion: "What should be done with, for, and to the freshmen having one unit of algebra?"

Professor Schoonmaker being absent due to an accident, her paper was read by Professor Anna D. Lewis.

Abstracts of the papers follow below, the number corresponding to the numbers in the list of titles:

1. Professor Anderson pointed out that the teacher of college mathematics is confronted with the problem of creating interest in case that interest has not been created in high school and of maintaining that interest in case it has already been created. Among factors which enter are the teacher, the subject matter, and the student himself. The preparation of the teacher should include a thorough course in physics, and a knowledge of astronomy, in order that he may the more fully appreciate the practical application of mathematics and thus be able to make it a more real, live subject to the student. He should likewise appreciate the application to the various fields of engineering.

Not only does the content of the course mean much, but in the freshman course especially, the sequence of subjects and the methods of their presentation as well as the amount of time devoted to each play an important rôle. Honor problems should stimulate the better students to greater endeavor with a consequent increase in interest. Honor courses, to which only the students having attained a high standing in mathematics are eligible, present a fine opportunity for the teacher to do work along some lines in which he is particularly interested.

2. (Owing to the widening interest of mathematicians in both the theory and the practice of statistics in business, Dr. Lincoln, who did his college work in Ohio and in Oxford as a Rhodes Scholar from Ohio, was invited to appear on the program.)

The need for study and practical application of business statistics was clearly demonstrated in the years immediately following the World War. Until 1920 prices had been rising for about 25 years and it was comparatively easy for concerns with indifferent management to make money. The recent industrial collapse, and the period of price uncertainty which followed, have caused leaders in business to think more seriously on economic questions. Hence the profession of "business economist" is gradually developing.

The "business economist" or "statistician," in order to do his work effectively, must master the facts of his business and must build upon the work of the accountant. He must take a comprehensive view of the various functions and departments of the organization and study their relations with a view to making

higher profits for the company or improving the service at the same or lower costs. Internal performance must be related to external business conditions. The study of one is ineffective without the study of the other. The principal job of the economist, therefore, is that of forecasting the future of business and the future of his own industry. This work necessitates the use of refined statistical methods combined with a wide economic background and the capacity to "think straight and see far." Generalizations are frequently unsafe; each industry must be studied in its peculiar relations to the general business movement. Prices, wages, interest rates and sales performance must always be thought of in terms of the future based on the experience of the past.

It is extremely difficult to apply definite quantitative measurements to the work of the business economist. Much that he does must necessarily be of a "developmental" sort and much of his work is in the nature of "insurance." Sometimes he may be able to save money directly, through giving sound advice at the proper time. More commonly, however, his chief function is to act as a balance wheel in "steadying" the administration policies in order to avoid losses through unscientific management. His job is to help level the peaks and fill up the valleys in business. Perhaps the test of the value of his service is the non-fulfillment of his prophecies, due to the fact that the concern which he advises by taking thought avoids the disasters which have been foreseen.

This general statement was followed by specific reference to the type of work which is being attempted in the General Statistical Department of the Western Electric Company.

3. In the usual discussions of the coefficient of correlation, this quantity is expressed by means of a formula in terms of n observed pairs of values of two variables x and y. In order to penetrate more deeply into the connection between the value of r and the degree of relationship between the variables, it is desirable to derive a formula which expresses r in terms of the underlying variables on which x and y themselves depend. If we assume that these variables are independent and that x and y are approximately linear functions of them, we obtain the formula $r = \Sigma a_{1n}a_{2n}s_n^2/\sqrt{\Sigma a_{1n}^2 s_n^2 \Sigma a_{2n}^2 s_n^2}$, where the a's are the coefficients in the linear expressions and the s's are the standard deviations of the independent variables. This formula was first given by Professor C. N. Moore in a paper in volume XLII of *Science*. In the present paper he makes use of the formula to set up certain simple examples of closely related variables having a coefficient of correlation equal to zero. In all the cases considered the regression is either exactly linear or very approximately so, and hence the ordinary interpretation of the correlation coefficient would lead to the conclusion that the variables x and y are unrelated.

4. In this discussion, Professor Reynolds first presented (in a more elaborate form) the suggestions he had published in the MONTHLY (*1922*, 122). He then suggested an arrangement of the elementary formulæ for insurance premiums, in two-dimensional array, according to the insurance protection afforded by the policy on the one hand, and according to the payment plan on the other. With

this arrangement of formulæ as a basis he suggested that students be instructed to break all premium problems up into problems in variation, the premium varying directly with a factor determined by the protection afforded by the policy and inversely with a factor determined by the payment plan adopted.

5. The purpose of Professor Armstrong's paper was to give some notion of how much actual use the investment houses make of the mathematics of finance in their ordinary work. The material came from illustrations accumulated during the past few years, and from recent inquiries directed to representative financial institutions. Requests were made for illustrations of actual practice in making up bids and selling prices, in deciding upon conditions for calling and refunding securities, in handling sinking funds, and in treating other problems of a mathematical nature. While these inquiries were answered carefully, very few actual illustrations were supplied.

Two examples of handling bond issues were presented in some detail. Conclusions were that mathematical theories find comparatively small employment in the routine of investment houses, their work being mostly confined to the use of bond tables; that the order in which merit would probably be rewarded is, ability to sell bonds, ability to forecast business conditions and the trend of the money market, ability to use the mathematics of finance.

The United States Treasury department furnished interesting material connected with government bonds, income tax, and "bonus bill" investigations.

6. The special quadratic $y = x^2 + bx + c$ denotes a unit parabola in (x, y) and a line in (b, c). If a point (x, y) traverse the parabola $y = px^2 + hx + k$, the corresponding lines envelop the parabola $c - k = [1/4(1 - p)](b - h)^2$. Reciprocally, if a point (b, c) traverse the parabola $c - k = [1/4(1 - p)](b - h)^2$, the corresponding parabolas envelop $y = px^2 + hx + k$.

In this correspondence *line* and *parabola* are duals while *point* corresponds to *point*. Thus the points of tangency mutually correspond. If $y = 0$, then $c = \frac{1}{4}b^2$ is the important special case of the discriminant.

7. Miss Schoonmaker's paper consisted first of a survey of the department of mathematics at the Western College for Women since 1915–16, the last year in which there was only one full-time instructor in the department. Next was a description of a method of sectionizing which was being tried out, and a statement of the results already obtained.

8. Professor Nordgaard reviewed briefly his experience with the grouping of freshmen on a scholarship basis at the University of Maine in 1915 and at Grinnell College in 1918–21; in the former the classification took place at the end of the first semester, and in the latter at the end of six weeks. He then described the plan he has used at Antioch College the last two years: On registration day every freshman takes a prognostic test, consisting of 25 to 30 short questions in arithmetic, algebra, and geometry, to be answered in an hour; based on the results, five or six groups are formed, each sufficiently homogeneous to have its own mode of presentation and to progress at its own pace. The schedule committee holds open for the mathematics department a block of class periods inside of which

students may shift without having a conflict with other studies. It was found necessary to shift only 5 per cent. during the semester.

To investigate the relative reliability of the prognostic test, all freshmen took the same class examination on the assignment covered by all the sections; between placement tests and uniform class tests the coefficient of correlation was .531 for the first year and .534 for the second year. A lower correlation obtained between the semester's class work and class tests taken at the end of five weeks.

Professor Nordgaard also brought out the correlation obtaining between the scores of the placement test and the semester ranks, the placement scores and the ratings of the Thurston intelligence tests, the Thurston ratings and the semester ranks. As a means of reliability he used the scale worked out by A. R. Crathorne in Chapter X of the *Report by the National Committee on Mathematical Requirements*.

9. The Section has six committees on the mathematical situation in Ohio. Four of these reported at length in 1923. Professor E. H. Clarke, chairman of the committee on college entrance requirement, presented some new material.

10. This subject for discussion was suggested by the new conditions presented due to lowering of entrance requirements in mathematics in many Ohio colleges, making it necessary for those institutions to adopt a policy as regards deficiencies and elementary mathematics. Professor Yanney, who proposed the subject, led the discussion. He opposes the colleges burdening themselves with courses for making up deficiencies. Professor Wiley is optimistic about further "experimenting" on the freshmen. At Denison they have been using student assistants for deficients with success. Professor Cairns intimated that Oberlin College might soon eliminate all applicants deficient in algebra. Professor Rasor suggested that mathematics teachers should not be too backward in "advertising" their studies. Professor Swartzel took an optimistic view of present conditions. Professor Wildermuth advocated more acquaintance with the principles of teaching if we wish to impress the leaders of education. Others contributed interesting suggestions. Miss Gertrude Silver, teacher in the North high school of Columbus, made an effective plea for the college teachers to study more sympathetically the problems of the high school pupils arising out of the high pressure of modern life, especially in the cities.

G. N. ARMSTRONG, *Secretary-Treasurer.*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION.

The third annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Georgia, at Athens, Georgia, on March 7–8, 1924. On Friday night a special dinner was given in honor of Professor H. E. Slaught with Professor R. P. Stephens presiding as toastmaster. There were about fifty present at this dinner including representatives of several departments of the University of Georgia as well as the members of the Association. All were delighted to honor the "Daddy of the Association" in this way. The following seventeen members were present: Eli Allison, E. A. Bailey, D. F. Barrow, S. M. Barton, T. R. Eagles, Floyd Field, Tomlinson Fort, Miss Leslie Gaylord, J. C. Hinton, J. W. Lasley, Jr., A. V. Martin, J. F. Messick, A. B. Morton, W. T. Peed, W. W. Rankin, Jr., H. E. Slaught, R. P. Stephens.

The following officers were elected for 1924–1925: Chairman, TOMLINSON FORT, University of Alabama; Vice-chairman, S. M. BARTON, University of the South; Secretary-treasurer, W. W. RANKIN, Jr., Agnes Scott College. Professor T. R. EAGLES and Miss LESLIE GAYLORD were appointed to act with the sec.-treas. as a Program Committee. The next meeting will be held at Birmingham, Alabama.

The following papers were presented:

(1) "The Association, its ideals, accomplishments and prospects" by Professor H. E. SLAUGHT.

(2) "Imaginary and infinite elements in undergraduate and high school teaching" by Professor TOMLINSON FORT.

(3) "Early teaching of mathematics at the University of Georgia" by Professor R. P. STEPHENS.

(4) "Mathematics and other sciences" by Professor H. E. SLAUGHT.

(5) "Some contacts of projective geometry with elementary mathematics" by Professor J. W. LASLEY, Jr.

(6) "The cultural value of mathematics" by Professor W. W. RANKIN, Jr.

(7) Discussion: "What can the Association do to aid individual departments" led by Professor TOMLINSON FORT.

Abstracts of the papers follow below, the numbers corresponding to the numbers given in the list of titles:

1. At the banquet given in his honor on Friday evening, Professor Slaught outlined the development of mathematics in America since the opening of Johns Hopkins University and showed how important a part had been played by the American Mathematical Society. He then set forth the rôle of the Mathematical Association of America and showed how much it had contributed in eight short years of its existence. After outlining the type of work done by each of these organizations, he called attention to the fine spirit of coöperation existing between the two, each in its own field carrying forward the development of mathematics and the teaching of mathematics.

He called upon all who are devotees of mathematics loyally to support both organizations in every way possible, calling attention to the strenuous condition

at the present time owing to the enormous increase in the cost of printing. He urged that so far as possible every one teaching mathematics in the colleges and universities should be a member of both the Society and the Association, especially as the combined annual dues in the two organizations are not as great as the dues in many other national scientific societies. Professor Slaught made a strong appeal and aroused the entire group to a finer and deeper interest in mathematics.

2. The discussion of Prof. Fort followed the lines indicated in a note published in this MONTHLY (*1923*, 255–256).

3. According to Professor Stephens, great emphasis was put upon mathematics in the early curriculum—one fourth of the students' entire time was supposed to be given to the courses in mathematics. The first entrance requirement in mathematics was arithmetic through the "rule of three" which was imposed about 1815. This was gradually raised till 1857 when all of arithmetic, elementary algebra through quadratics, and three books of Legendre were required. The first college course adopted in 1801 embraced arithmetic, algebra, trigonometry with many applications to surveying and navigation, and astronomy. The scope of the requirement in mathematics was gradually enlarged till in 1850 both analytic geometry and calculus were included in the required curriculum.

4. Professor Slaught spoke briefly but straight to the point. He pointed out the fundamental relation which exists between mathematics and such sciences as biology, chemistry, physics, sociology and others. He further suggested that a subject was only entitled to call itself a science when the knowledge of this subject could be reduced to mathematical formulæ.

5. Practically all of the advanced students of mathematics, Professor Lasley maintained, are training to be teachers of the subject. The various fields of elementary mathematics were taken up in turn and in detail some of the topics on which projective geometry throws light were pointed out. The analytical phase of the subject was stressed as a necessary supplement to the geometric. Mr. Lasley claimed for that subject an integral part in the training for teachers of elementary mathematics. It has been too long regarded as merely one of the avenues to graduate mathematics.

6. Professor Rankin defined culture as follows: Culture is an intelligent interest in the past, present, and future achievements of man. An intelligent interest in an achievement was explained to mean some knowledge of the history of the achievement, some knowledge of the underlying principles and laws which have assisted in making the achievement, and some knowledge of the value of the achievement. The definition of mathematics used was that of Benjamin Peirce —"Mathematics is the science which draws necessary conclusions." Professor Rankin based the claim for cultural value of mathematics on these two definitions and an intelligent interest in the following widely diversified list of achievements of man: Number system, language, process of thinking, art, architecture, music, protestant reformation, transportation and communication, banking and commerce, medicine, political economy and sociology. He showed that each of these achievements was in some way related to mathematics.

W. W. RANKIN, Jr., *Secretary-Treasurer.*

INTEGRAL INEQUALITIES WITH APPLICATIONS TO THE CALCULUS OF VARIATIONS.

By OTTO DUNKEL, Washington University.

1. Introduction. Two sets of integral inequalities will be developed in this paper which apply directly to the solution of certain problems of the calculus of variations. These problems belong, as might be expected, to the simplest cases which present themselves in that study. The examples in the calculus of variations which are considered are not the only ones which may be treated by these inequalities. They include, however, the two problems which have been treated before by the writer. When auxiliary conditions are introduced the inequalities do not appear to suffice except in certain special types to which the second set of inequalities apply. A more general case with auxiliary conditions is considered at the end of the paper. The method of treatment is uniform throughout the discussion. The first set of inequalities may be readily thrown into a form which exhibits the relation between the mean values of functions as expressed by definite integrals. The second set of inequalities give a necessary and sufficient condition for the linear independence of a set of functions of one variable. If the integrals are replaced by finite sums, corresponding inequalities and theorems may be obtained by the same methods used here.

All of the functions which are considered are assumed to be single-valued and continuous in a single independent variable in the intervals considered.

2. The First Set of Integral Inequalities. We shall consider first a known inequality theorem and derive from it a sequence of inequalities. The proof which will be given will show that the usual statement of this theorem, which follows, requires more than is necessary to establish the inequality.

THEOREM. *If $f_1(x)$ and $f_2(x)$ are two functions of x, neither of which is a constant, such that if either one of them experiences an increase in any finite interval included in the interval of integration the other experiences a decrease, and conversely, then*

$$\int_a^b f_1(x)dx \int_a^b f_2(x)dx > (b-a)\int_a^b f_1(x)f_2(x)dx. \tag{1}$$

If on the other hand the two functions experience increments in the same intervals and decrements in the same intervals, the inequality sign above is reversed.[1]

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[1] The theorem as stated may be deduced directly from the identity

$$\left(\sum_1^n a_i\right)\left(\sum_1^n b_j\right) = n\sum_1^n a_ib_i + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i - a_j)(b_j - b_i),$$

which gives

$$\int_a^b f_1(x)dx \int_a^b f_2(x)dx = (b-a)\int_a^b f_1(x)f_2(x)dx + \frac{1}{2}\int_a^b\int_a^b [f_1(x) - f_1(y)][f_2(y) - f_2(x)]dxdy.$$

From this result it will be seen that a proof without the use of the algebraic identity may be ob-

PROOF. Since $f_1(x)$ is continuous, we may set

$$\int_a^b f_1(x)dx = f_1(c)(b-a), \qquad a < c < b, \tag{2}$$

and then

$$f_1(x) = f_1(c) + \eta_1(x), \qquad f_2(x) = f_2(c) - \eta_2(x). \tag{3}$$

From (2) and (3) result

$$\int_a^b \eta_1(x)dx = 0, \qquad \int_a^b \eta_1(x)f_2(x)dx = -\int_a^b \eta_1(x)\eta_2(x)dx. \tag{4}$$

It now follows from (2), (3) and (4) that

$$\begin{aligned}\int_a^b f_1(x)dx \int_a^b f_2(x)dx &= (b-a)\int_a^b f_1(c)f_2(x)dx, \\ &= (b-a)\int_a^b f_1(x)f_2(x)dx + (b-a)\int_a^b \eta_1(x)\eta_2(x)dx. \end{aligned} \tag{5}$$

By hypothesis $\eta_1(x)\eta_2(x)$ can never be negative. Since $f_1(x)$ is not a constant, $\eta_1(x)$ must be different from zero for some value of x, say $\bar{x}$, and by hypothesis $\eta_2(x)$ must also be different from zero and of the same sign. Hence $\eta_1(x)\eta_2(x)$ must be greater than zero in an interval about $\bar{x}$, since this product is continuous. It follows then that the last integral to the right in (5) is surely greater than zero, and therefore (1) must be true.

The second part of the theorem follows in the same way from (5), observing in this case that $\eta_1(x)\eta_2(x)$ is never positive, and that it must be less than zero in some interval within the interval of integration.

REMARKS. In this proof we need only the requirement that for one determination of c from either $f_1(x)$ or $f_2(x)$ the product $\eta_1(x)\eta_2(x)$ shall never be negative, in the case of the first part of the theorem; and for at least one value of x it shall be greater than zero. It is possible for two functions to satisfy this demand without satisfying the original requirement. Thus $f_1(x) = 1 - x$, $f_2(x) = 16x - x^2$, in the interval $0 \leqq x \leqq 9$, is such an example, where $c = 4.5$. Here $\eta_1(x)\eta_2(x) > 0$ if $x \neq 4.5$. From 0 to 8, $f_2(x)$ increases and $f_1(x)$ decreases, while from 8 to 9 both decrease.

Corresponding theorems in regard to double integrals may be easily stated and proved in the same way, where $f_1(x)$ and $f_2(x)$ are replaced by functions of two variables.

3. Special Inequalities of the First Set. From the above theorem follow special cases of the inequalities which will be used later and which we proceed to state.

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tained by starting with the double integral on the right and proceeding as in Goursat-Hedrick's *Mathematical Analysis*, vol. 1, page 257. See also the proof of the special case where $b_i = a_i^{-1}$ in this journal [*1924*, 148–150].

THEOREM. *If $f_1(x), f_2(x), \cdots, f_n(x)$ are never negative and experience simultaneously increments, and decrements, then*

$$\int_a^b f_1(x)dx \int_a^b f_2(x)dx \cdots \int_a^b f_n(x)dx < (b-a)^{n-1} \int_a^b f_1(x)f_2(x)\cdots f_n(x)dx. \quad (6)$$

In particular, if they are all equal,

$$\left[\int_a^b f(x)dx\right]^n < (b-a)^{n-1} \int_a^b [f(x)]^n dx. \quad (7)$$

This follows at once from the second part of (1) since the product of any pair of these functions varies in the same manner as any one of them, and hence such a product may be used as one of the functions in (1).

REMARKS. If n is even, the requirement that $f(x)$ shall never be negative in (7) may be dropped. For the second part of (1) is true if $f_1(x) = f_2(x) = f(x)$ without any such restriction upon $f(x)$. Thus (7) is true in this case if $n = 2$. If n is even and more than 2 we have merely to combine the $n/2$ relations (7) in which $n = 2$ by a repeated application of the second part of (1).

THEOREM. *If $f(x) > 0$ and n is a positive integer,*

$$\left[\int_a^b \frac{dx}{f(x)}\right]^n \int_a^b [f(x)]^n dx > (b-a)^{n+1}. \quad (8)$$

PROOF. Since $f(x) > 0$, the two functions $[f(x)]^n$ and $1/f(x)$ satisfy the conditions for the first part of (1) without further restrictions upon $f(x)$. Hence

$$\int_a^b [f(x)]^n dx \int_a^b \frac{dx}{f(x)} > (b-a) \int_a^b [f(x)]^{n-1} dx. \quad (9)$$

If we multiply both sides by $\int_a^b [f(x)]^{-1} dx$ and apply to the resulting right side the above inequality (9) in which n is replaced by $n-1$, and continue in this way, we obtain finally the inequality (8).

REMARKS. In the inequalities (7) and (8) it has been assumed that $f(x)$ is not a constant. If it is a constant, then the equality sign must be used instead of the inequality sign.

In these results the exponent n is a positive integer, but these inequalities are also true within certain limitations when the exponent is any real number. This will be shown by considering next a general theorem.

4. The Generalized Inequality Theorem. Let $F(t)$ be a function of t which has a second derivative $F''(t)$ which does not change sign and which vanishes, if at all, only at isolated points of the interval $t_1 \leqq t \leqq t_2$, where $t_1 \leqq f(x) \leqq t_2$, $a \leqq x \leqq b$. We may state the following:

THEOREM.

$$\frac{\int_a^b F[f(x)]dx}{b-a} \gtreqless F\left[\frac{\int_a^b f(x)dx}{b-a}\right] \text{ according as } \begin{matrix} F''(t) \geqq 0, \\ \text{or} \\ F''(t) \leqq 0. \end{matrix} \quad (10)$$

PROOF. Setting, as before in (2) and (3), $f(x) = m + \eta$, where for brevity we write $f(c) = m$ and $\eta(x) = \eta$, we have

$$F[f(x)] = F(m + \eta) = F(m) + \eta F'(m) + \frac{\eta^2}{2} F''[m + \theta\eta], \qquad 0 < \theta < 1. \quad (11)$$

If $\eta \neq 0$ for any value of x, then $F''[m + \theta\eta] \neq 0$; for if it is zero, then the function of t, $F[m + t] - F(m) - tF'(m)$ is zero for $t = 0$ and $t = \eta$. Hence $F'[m + t] - F'(m)$ must vanish for some value of t, say $\bar{t}$, between 0 and η. Thus $F'[m + \bar{t}] = F'(m)$ and this cannot be true since $F'[m + t]$ always increases or always decreases in the interval for t.

Integrating (11), and noting that the integral of $\eta F'(m)$ is zero, we have

$$\int_a^b F[f(x)]dx = (b - a)F(m) + \frac{1}{2}\int_a^b \eta^2 F''[m + \theta\eta]dx. \quad (12)$$

Since $\eta(x)$ is not identically zero, the integrand on the right is greater than zero for some values of x if $F''(t) \geqq 0$, less than zero if $F''(t) \leqq 0$. Hence the theorem follows.

REMARKS. This result may be put in a symmetric form. Let $\varphi(t)$ and $\psi(t)$ be functions of t which have first derivatives $\varphi'(t)$ and $\psi'(t)$ which do not change sign and which vanish at only isolated points. It will be convenient to suppose that $\psi'(t)$ is never negative. Denote the inverse of the two functions by $\varphi^{-1}(t)$ and $\psi^{-1}(t)$. Replace $f(x)$ in (10) by $\varphi[f(x)]$ and $F(t)$ by $\psi[\varphi^{-1}(t)]$. Then (10) becomes after a slight change

$$\psi^{-1}\left[\frac{\int_a^b \psi[f(x)]dx}{b - a}\right] \gtrless \varphi^{-1}\left[\frac{\int_a^b \varphi[f(x)]dx}{b - a}\right], \quad (10')$$

where the upper sign is used if the second derivative of $\psi[\varphi^{-1}(t)]$ is never negative, and the lower sign if it is never positive.[1]

It is now possible to extend the results in (7) and (8) to any real exponent. If in (10) we set $F(t) = t^\alpha$, $\alpha \neq 1$, then

$$(b - a)^{\alpha-1}\int_a^b [f(x)]^\alpha dx > \left[\int_a^b f(x)dx\right]^\alpha, \qquad \alpha = 2 + \frac{2j}{2i + 1}, \quad (13)$$

where i and j are integers greater than or equal to zero and where $f(x)$ may change sign.

If $f(x) > 0$, *then*

$$\begin{aligned}(b - a)^{\alpha-1}\int_a^b [f(x)]^\alpha dx &> \left[\int_a^b f(x)dx\right]^\alpha, \qquad \alpha < 0 \quad \text{or} \quad 1 < \alpha, \\ &< \left[\int_a^b f(x)dx\right]^\alpha, \qquad 0 < \alpha < 1.\end{aligned} \quad (14)$$

[1] This result is analogous to the theorem contained in the paper by the writer "Generalized Geometric Means and Algebraic Equations," *Annals of Mathematics*, 2d ser., vol. 11, 1909, page 26.

Except in the case of $\alpha < 0$, a simple consideration will show that (14) is true even when $f(x)$ is allowed to vanish in the interval of integration, but not becoming negative. [1]

If we replace $f(x)$ by $1/f(x)$ and α by $-\alpha$, the inequalities (14) may be written

$$\left[\int_a^b \frac{dx}{f(x)}\right]^\alpha \int_a^b [f(x)]^\alpha dx > (b-a)^{\alpha+1}, \qquad \alpha > 0, \quad \text{or} \quad \alpha < -1, \tag{14'}$$
$$< (b-a)^{\alpha+1}, \qquad -1 < \alpha < 0,$$

where $f(x) > 0$.

5. The Second Set of Integral Inequalities. This second set of integral inequalities is of a more complicated form, but it will be shown that it includes special cases of the first set. Here n functions, $f_1(x), f_2(x), \cdots, f_n(x)$, will be considered which are linearly independent in the interval $a \leqq x \leqq b$. This means that there do not exist n constants, $A_1, A_2, \cdots, A_n$, not all zero, such that

$$\sum_1^n A_i f_i(x) = 0, \qquad a \leqq x \leqq b.$$

If, however, there does exist such a set of constants, the n functions are said to be linearly dependent in the given interval. For brevity of writing we shall set

$$\int_a^b f_i(x) f_j(x) dx = (f_i f_j) = (f_j f_i),$$

and

$$\begin{vmatrix} (f_1f_1) & (f_1f_2) & \cdots & (f_1f_n) \\ (f_2f_1) & (f_2f_2) & \cdots & (f_2f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_nf_1) & (f_nf_2) & \cdots & (f_nf_n) \end{vmatrix} = |f_1 f_2 \cdots f_n|. \tag{15}$$

We shall prove the following

THEOREM. *If the n functions are linearly independent in the given interval, the determinant* (15) *is greater than zero. If they are linearly dependent in the interval, the determinant is zero.*

PROOF. The second part of the theorem is obvious. For in this case either one column consists of zeros, or is a linear combination of the other columns. We turn now to the first part of the theorem. Since the n functions are linearly independent, any smaller set selected from them must also be linearly independent. Suppose then that the theorem is true for $f_i(x), f_{i+1}(x), \cdots, f_n(x)$; it

[1] Certain known inequalities are contained in the above as also in (10'). For example, if $a = 0, b = 1, f(x) = x^q, \alpha = p/q, p > q > 0$, it follows from (14) that

$$(1+q)^{1/q} > (1+p)^{1/p}.$$

will then be possible to determine $n - i + 1$ constants, $A_i, A_{i+1}, \cdots, A_n$, such that

$$\begin{aligned}
(f_i f_{i-1}) &= A_i(f_i f_i) + A_{i+1}(f_i f_{i+1}) + \cdots + A_n(f_i f_n), \\
(f_{i+1} f_{i-1}) &= A_i(f_{i+1} f_i) + A_{i+1}(f_{i+1} f_{i+1}) + \cdots + A_n(f_{i+1} f_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(f_n f_{i-1}) &= A_i(f_n f_i) + A_{i+1}(f_n f_{i+1}) + \cdots + A_n(f_n f_n),
\end{aligned} \tag{16}$$

since the determinant $|f_i f_{i+1} \cdots f_n|$ of the equations is greater than zero by hypothesis. Set

$$f_{i-1}(x) = A_i f_i(x) + A_{i+1} f_{i+1}(x) + \cdots + A_n f_n(x) + \eta_{i-1}(x). \tag{17}$$

The function $\eta_{i-1}(x)$ cannot be identically zero in the given interval; for if it were, then the $n - i + 2$ functions would be linearly dependent, as also the whole set of n functions. Multiplying (17) by $f_i(x), f_{i+1}(x), \cdots, f_n(x)$ in turn and integrating each result, we find by (16)

$$(\eta_{i-1} f_j) = 0, \quad j = i, \quad i+1, \quad \cdots, \quad n. \tag{18}$$

Multiplying (17) by $\eta_{i-1}(x)$ and integrating, we obtain by use of (18)

$$(f_{i-1}\eta_{i-1}) = (\eta_{i-1}\eta_{i-1}). \tag{18'}$$

In the same manner by multiplying (17) by $f_{i-1}(x)$, integrating, and then using (18'), we obtain

$$(f_{i-1} f_{i-1}) - (\eta_{i-1}\eta_{i-1}) = A_i(f_{i-1} f_i) + A_{i+1}(f_{i-1} f_{i+1}) + \cdots + A_n(f_{i-1} f_n). \tag{19}$$

Now this equation with (16) gives a set of $n - i + 2$ equations which admit the solution $-1, A_i, A_{i+1}, \cdots, A_n$. Hence their determinant must be zero. This determinant breaks up into two parts, and we obtain the equation

$$|f_{i-1} f_i f_{i+1} \cdots f_n| = (\eta_{i-1}\eta_{i-1})\,|f_i f_{i+1} \cdots f_n|. \tag{20}$$

The factor $(\eta_{i-1}\eta_{i-1})$ is greater than zero; also $|f_i f_{i+1} \cdots f_n|$ is greater than zero by hypothesis; and hence the determinant on the left is also greater than zero. If $i = n$, $|f_n| = (f_n f_n)$ and (20) becomes

$$|f_{n-1} f_n| = (\eta_{n-1}\eta_{n-1})(f_n f_n).$$

Since the n functions are linearly independent no one of them can be identically zero and hence $(f_n f_n) > 0$. Then the proof above shows that $|f_{n-1} f_n|$ is greater than zero, and so on; and we find at last that the determinant in (15) is greater than zero.

REMARKS. If $f(x)$ is not a constant, then $f(x)$ and 1 are linearly independent. Hence

$$\begin{vmatrix} (ff) & (1f) \\ (f1) & (11) \end{vmatrix} > 0, \quad \text{or} \quad (b-a)\int_a^b [f(x)]^2 dx > \left[\int_a^b f(x)dx\right]^2.$$

If $f(x) > 0$ and is not a constant, then $\sqrt{f(x)}$ and $1/\sqrt{f(x)}$ are linearly independent and we find in the same way

$$\int_a^b \frac{dx}{f(x)} \int_a^b f(x)dx > (b-a)^2.$$

Thus this second set of inequalities leads to special cases of the first set. By considering powers of $f(x)$, other inequalities may be obtained.

If both sides of (17) are squared and integrated, there results by use of (18)

$$(f_{i-1}f_{i-1}) - (\eta_{i-1}\eta_{i-1}) = \int_a^b \left[\sum_i^n A_j f_j(x)\right]^2 dx. \tag{19'}$$

The integral on the right is zero only if $A_i = A_{i+1} = \cdots = A_n = 0$, and then by (16) $(f_{i-1}f_j) = 0$, $j = i, i+1, \cdots, n$, and conversely. In this case $f_{i-1}(x)$ is said to be orthogonal to the functions with the following subscripts. Hence from (20) there follows

$$|f_{i-1}f_i f_{i+1} \cdots f_n| \leqq (f_{i-1}f_{i-1})\,|f_i f_{i+1} \cdots f_n|, \tag{20'}$$

where the equality sign is used if $f_{i-1}(x)$ is orthogonal to each of the other functions appearing in this relation. By a repeated application of (20') there results the

THEOREM.

$$|f_1 f_2 \cdots f_n| \leqq (f_1 f_1)(f_2 f_2) \cdots (f_n f_n),$$

where equality occurs only if every pair of different functions is an orthogonal pair.

6. The Minimum of Certain Definite Integrals. The previous results will now be applied to the determination of the minimum of certain types of definite integrals.

CASE 1. Let y be a function of x which takes on the values 0 and y_2, respectively, at the ends of the interval $0 \leqq x \leqq x_2$, and which possesses a derivative y' which is greater than zero in this interval. Also let $\varphi(y)$ be a function of y which is greater than zero for all the values of y which are considered. It is desired to find that function y of x which renders

$$\int_0^{x_2} \frac{\varphi(y)}{y'^\alpha}\,dx, \qquad \alpha > 0, \quad \text{or} \quad \alpha < -1, \tag{21}$$

a minimum. Using (14') we have

$$\left[\int_0^{x_2} \frac{y'dx}{[\varphi(y)]^{1/\alpha}}\right]^\alpha \int_0^{x_2} \frac{\varphi(y)dx}{y'^\alpha} > x_2^{\alpha+1}. \tag{22}$$

The integral in the first factor on the left is a constant since it is equal to

$$\int_0^{y_2} \frac{dy}{[\varphi(y)]^{1/\alpha}}.$$

Since in (22) equality exists only when the integrand is a constant, the minimum of (21) is given [1] by

$$\frac{\varphi(y)}{y'^{\alpha}} = A, \qquad x_2{}^{\alpha} = A\left[\int_0^{y_2} \frac{dy}{[\varphi(y)]^{1/\alpha}}\right]^{\alpha}, \qquad x = A^{1/\alpha}\int_0^{y} \frac{dy}{[\varphi(y)]^{1/\alpha}}. \tag{23}$$

CASE 2. If the integral to be minimized is of the form

$$\int_0^{x_2} \frac{\varphi(x)dx}{y'^{\alpha}}, \qquad \alpha > 0, \quad \text{or} \quad \alpha < -1, \tag{24}$$

where $\varphi(x)$ is greater than zero, it will be convenient to write it

$$\int_0^{y_2} \varphi(x)\left[\frac{dx}{dy}\right]^{\alpha+1} dy, \tag{24'}$$

and to consider x as a function of y. Using (14) we have

$$\int_0^{y_2} \varphi(x)\left(\frac{dx}{dy}\right)^{\alpha+1} dy > \frac{1}{y_2{}^{\alpha}}\left[\int_0^{x_2} [\varphi(x)]^{1/\alpha+1}dx\right]^{\alpha+1}. \tag{25}$$

We have a minimum only when the integrand on the left is a constant. Hence

$$\varphi(x)\left(\frac{dx}{dy}\right)^{\alpha+1} = A, \qquad A^{1/\alpha+1}y = \int_0^{x} [\varphi(x)]^{1/\alpha+1}dx, \tag{26}$$

where A is determined by the fact that $y = y_2$ when $x = x_2$.

CASE 3. Let us consider now the case in which y is zero when x is zero, its derivative y' takes on the values p_1 and p_2, $-\infty < p_1 < p_2 < +\infty$, at $x = 0$ and $x = x_2$, respectively, and the second derivative y'' exists and is greater than zero in the interval for x. It is desired to determine the function y so as to render the integral

$$\int_0^{x_2} \frac{\varphi(y')dx}{y''^{\alpha}}, \qquad \alpha > 0, \quad \text{or} \quad \alpha < -1, \tag{27}$$

a minimum, where $\varphi(y') > 0$.

As before we have from (14')

$$\left[\int_0^{x_2} \frac{y''dx}{[\varphi(y')]^{1/\alpha}}\right]^{\alpha}\int_0^{x_2} \frac{\varphi(y')dx}{y''^{\alpha}} > x_2{}^{\alpha+1},$$

where the integral in the first factor on the left is a constant, since it is equal to

$$\int_{p_1}^{p_2} \frac{dp}{[\varphi(p)]^{1/\alpha}}.$$

Hence the minimum is given by setting the integrand in (27) equal to a constant A. We have then

$$x = A^{1/\alpha}\int_{p_1}^{p} \frac{dp}{[\varphi(p)]^{1/\alpha}}, \qquad y = A^{1/\alpha}\int_{p_1}^{p} \frac{pdp}{[\varphi(p)]^{1/\alpha}}, \tag{28}$$

[1] The case in which $\varphi(y) = 1$ and $\alpha = -2$ is treated in this same manner in Goursat-Hedrick's *Mathematical Analysis*, vol. 1, pp. 257–258.

where A is determined by the fact that $x = x_2$ when $p = p_2$; the second integral determines the value of y when $x = x_2$.

REMARKS. If instead of fixing the value of x_2 as above, the value of y_2 is given and $y' = p_2$ when $y = y_2$ while the other conditions remain the same, the integral (27) may be written

$$\int_0^{y_2} \frac{\varphi(y')dy}{y''^{\alpha}y'}. \tag{27'}$$

Here $\varphi(y')/y'$ is to be continuous and greater than zero. Applying (14') we have

$$\left[\int_0^{y_2}\left[\frac{y'}{\varphi(y')}\right]^{1/\alpha} y''dy\right]^{\alpha}\int_0^{y_2}\frac{\varphi(y')dy}{y''^{\alpha}y'} > y_2^{\alpha_2+1}.$$

The integral in the first factor on the left is equal to

$$\int_{p_1}^{p_2}\left[\frac{p}{\varphi(p)}\right]^{1/\alpha} pdp,$$

and is therefore a constant. Hence the minimum of (27') is given by

$$\frac{\varphi(y')}{y''^{\alpha}} = By'. \tag{29}$$

These two problems are included in a more general one. Suppose that (27) is to be made a minimum by a function of x, $y = y(x)$, such that the curve defined by this equation passes through the origin with the slope p_1, and cuts the line $x\cos\gamma + y\sin\gamma = d$ at a given angle. By rotating the axes through the angle γ, this problem is reduced to one in which the integral is of the same form as (27). The conditions are to be made of the same nature as those given for that integral. The minimum is given by

$$\frac{\varphi(y')}{y''^{\alpha}} = C[\cos\gamma + y'\sin\gamma]. \tag{30}$$

If the values of x_2 and y_2 are both assigned, the preceding method does not seem to apply, but the results in the three cases considered would suggest that the solution is of the form

$$\frac{\varphi(y')}{y''^{\alpha}} = A + By', \tag{31}$$

where A and B are constants to be determined by the end conditions. If $\alpha = 1$, the second set of inequalities gives a proof of this, after changing the form of (27) and introducing auxiliary integral conditions. This case will now be considered.

7. Auxiliary Integral Conditions. The integral (27) will be written

$$\int_{p_1}^{p_2}\varphi(p)\delta^2 dp, \qquad \delta = \frac{dx}{dp}, \qquad \alpha = 1, \tag{32}$$

and we have the auxiliary conditions

$$x_2 = \int_{p_1}^{p_2} \delta dp, \qquad y_2 = \int_{p_1}^{p_2} \delta p dp. \tag{32'}$$

Here δ may be permitted to change sign. The two functions $1/\sqrt{\varphi(p)}$, $p/\sqrt{\varphi(p)}$ are continuous since we have assumed that $\varphi(p) > 0$, and they are linearly independent. Suppose that δ is any function of p which satisfies (32') and that $\delta\sqrt{\varphi(p)}$ is linearly independent of the above two functions. Then by the theorem for linear independence, page 330, we have

$$\left| \delta\sqrt{\varphi(p)}, \quad \frac{1}{\sqrt{\varphi(p)}}, \quad \frac{p}{\sqrt{\varphi(p)}} \right| > 0, \tag{33}$$

where the expression on the left is defined in (15). In the determinant on the left of (33) we shall have the following integrals:

$$(\delta\sqrt{\varphi(p)}\ \delta\sqrt{\varphi(p)}) = \int_{p_1}^{p_2} \varphi(p)\delta^2 dp,$$

$$\left(\delta\sqrt{\varphi(p)}\frac{1}{\sqrt{\varphi(p)}}\right) = \int_{p_1}^{p_2} \delta dp = x_2, \qquad \left(\delta\sqrt{\varphi(p)}\frac{p}{\sqrt{\varphi(p)}}\right) = \int_{p_1}^{p_2} \delta p dp = y_2.$$

Inserting these in (33) the inequality may be written in the form

$$D\int_{p_1}^{p_2} \varphi(p)\delta^2 dp > -\begin{vmatrix} 0 & x_2 & y_2 \\ x_2 & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \\ y_2 & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \end{vmatrix} = -\Delta, \tag{34}$$

where D is the cofactor of the element 0 in the determinant on the right, and it is a constant greater than zero since $1/\sqrt{\varphi(p)}$ and $p/\sqrt{\varphi(p)}$ are linearly independent. Also the determinant on the right is a constant. Equality exists in (34) when and only when $\delta\sqrt{\varphi(p)}$ is linearly dependent upon the above two functions. Hence the minimum of (32) is given by

$$\delta = \frac{A + Bp}{\varphi(p)}. \tag{35}$$

It is easily verified that

$$\begin{vmatrix} \varphi(p)\delta & 1 & p \\ x_2 & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{1}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \\ y_2 & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{1}{\sqrt{\varphi(p)}}\right) & \left(\frac{p}{\sqrt{\varphi(p)}}\frac{p}{\sqrt{\varphi(p)}}\right) \end{vmatrix} = 0. \tag{36}$$

The minimum value of the integral is $-\Delta/D = Ax_2 + By_2$. If the subscripts

2 in (32′) are omitted and the value of δ in (35) or (36) is inserted, the solution of this problem will be complete.[1]

Other independent conditions may be added to (32′); for example, it may be required that

$$l = \int_{p_1}^{p_2} \sqrt{1 + p^2}\delta dp,$$

where l is a given quantity. The solution may be obtained in the same way.

8. A More General Problem. Turning now to a more general case, suppose that it be required to determine δ as a function of p which renders

$$\int_{p_1}^{p_2} \varphi(p)F(\delta)dp, \tag{37}$$

a minimum, where $F(t)$ has a second derivative $F''(t)$ which is greater than zero, and where δ satisfies the conditions (32′). Let δ and $\delta + \eta$ be two different functions which satisfy the conditions (32′). Then

$$\int_{p_1}^{p_2} \eta dp = 0, \qquad \int_{p_1}^{p_2} \eta p dp = 0. \tag{38}$$

Also

$$\int_{p_1}^{p_2} \varphi(p)F(\delta + \eta)dp = \int_{p_1}^{p_2} \varphi(p)F(\delta)dp + \int_{p_1}^{p_2} \eta\varphi(p)F'(\delta)dp + \frac{1}{2}\int_{p_1}^{p_2} \eta^2\varphi(p)F''(\delta + \theta\eta)dp, \qquad 0 < \theta < 1. \tag{39}$$

Hence if δ is chosen so that

$$F'(\delta) = \frac{A + Bp}{\varphi(p)}, \tag{40}$$

where A and B are constants, then by use of (38) we shall have

$$\int_{p_1}^{p_2} \varphi(p)F(\delta + \eta)dp > \int_{p_1}^{p_2} \varphi(p)F(\delta)dp. \tag{39}$$

The constants A and B must be so chosen that the two equations in (32′) are satisfied. Since $F'(t)$ is an increasing function of t, the equation (40) admits a continuous solution for δ. In the case treated above where $F(t) = t^2$, there was no difficulty in the actual determination of the constants; but in other cases there may be difficulty not only in the actual determination of the constants but in knowing even if there exist values of A and B which make δ satisfy (32′). If there exists a set of values there can be only one set.

[1] For particular cases see the papers by the writer, "A Determination of the Curve Minimizing the Area Enclosed by it and its Evolute," in the MONTHLY [*1921*, 15–19]; "A Direct Determination of the Minimum Area between a Curve and its Caustic," *Annals of Mathematics*, 2d ser., vol. 23, no. 2, pp. 135–140, Dec. 1921. Also the paper in the same journal by Rider, "On the Minimizing of a Class of Definite Integrals," 2d ser., vol. 24, no. 2, pp. 167–174, Dec. 1922.

REMARKS. If $F(t) = t^{\alpha+1}$, $\alpha = [2(i+j)+1]/(2i+1)$, where i and j are integers, $j \geqq 0$, $i \geqq 0$, then it may be shown that there exists a unique set of constants A and B of the kind desired.

A SYSTEM OF TRIANGLES RELATED TO A PORISTIC SYSTEM.

By J. H. WEAVER, Ohio State University.

1. Introduction. In the last half century considerable attention has been paid to poristic systems of triangles, *e.g.*, triangles which are inscribed in one circle and circumscribed to another. In this paper it is proposed to set forth some of the simpler properties of a set of triangles having a fixed circumcircle and a fixed nine-point circle.

In this discussion several terms will be used which may not be familiar to the general reader. We proceed to define these terms.

1. *Orthopole:* An orthopole is defined by J. Neuberg as follows. Let there be a triangle ABC and a line l in the plane of the triangle. From A, B, and C draw perpendiculars to l cutting it in P, Q, R respectively. From P draw a perpendicular to BC, from Q a perpendicular to AC and from R a perpendicular to AB. These three perpendiculars meet in a point S which is called the orthopole of l with respect to the triangle ABC. It seems wise to prove [1] the property of the orthopole used in theorem VII below.

LEMMA: To determine the orthopole of l geometrically.

Let AP meet the circle ABC again in K. Draw the chord $K\,K'K''$ perpendicular to BC, cutting BC in K' and the circle again in K''. Let BA and CA meet l in R' and R'' respectively.

Let l make angles θ_1, θ_2, θ_3 with BC, CA, AB respectively. We then have $BK = 2R \sin (BAK \text{ or } R'AP) = 2R \cos \theta_3$ (R = radius of circle ABC). Also $CK = 2R \cos \theta_2$.

Therefore $KK' = BK \cdot CK/2R = 2R \cos \theta_2 \cos \theta_3 = SP$. Hence S is found by drawing $K'S$, PS parallels to AP, KK' respectively.

THEOREM: *The Simson lines of the extremities of any chord TT' of the circle ABC pass through S the orthopole of TT'.*

If we draw TT_1 cutting BC in X and perpendicular to BC and draw KK_1 perpendicular to TT_1 and cutting it in K_1, we have $\angle TPK = \text{rt } \angle = \angle TK_1K$. Therefore TK_1PR are cyclic, and $\angle PK_1X$ or $PK_1T_1 = \angle TKP$, or $\angle TKA = \angle TT_1A$. Hence K_1P is parallel to AT_1. But $K_1X = KK' = SP$. Therefore XS is parallel to K_1P and to T_1A and is the Simson line of T. And similarly for T'.

2. *Antipedal triangle:* Let S be any point in the plane of the triangle ABC. Draw SA, SB, SC and through A, B, C draw lines perpendicular to SA, SB, SC. These perpendiculars determine a triangle called the antipedal triangle of S.

[1] This proof together with other interesting proofs of various properties of the orthopole are due to J. Neuberg and may be found in Gallatley, *Modern Geometry of the Triangle*, Chap. VI.

3. *Antimedial Triangle:* The antimedial triangle is a triangle formed by drawing through A, B, C lines parallel to the opposite sides.

4. *Nagel Point:* The Nagel point is the point of intersection of the lines AX_1, BY_2, CZ_3, where X_1 is the point of contact on BC of the ex-circle opposite A etc.

2. Theorems. THEOREM I: *If a variable triangle has a fixed nine-point circle and a fixed circumcircle, it envelops a conic.*

PROOF: Let (O) be the circumcircle, center O, and (N) the nine-point circle, center N. Let ABC be the variable triangle. Draw the altitude CF and extend it to K on (O). Draw KO cutting AB in L. Let H be the ortho-center of ABC. Draw LH.

Then $HF = FK$ and LF is perpendicular to HK. Therefore the triangle LHK is isosceles and $LH = KL$. Therefore $OL + HL = R$ (R is radius of (O)). Also AB bisects $\angle KLH$. Hence L is on an ellipse (E) with N as center and H and O as foci.

REMARKS: This proof assumes that the circles (O) and (N) do not intersect. If they intersect, the conic will be a hyperbola and the triangle ABC will be obtuse angled.

The circumcenter, orthocenter and nine-point center are fixed points in this system, but the in-center and the ex-centers are variable. We proceed to find the locus of these points.

THEOREM II: *The in-center and the ex-centers lie on a quartic curve consisting of two ovals each of which is the inverse of the other with respect to N as center.*

PROOF: Let I be the in-center and let $OI = d$, $ON = c$. Let O be the origin, ON the X-axis and $I = (x, y)$. Then

$$x^2 + y^2 = d^2 = R^2 - 2Rr, \text{ by Euler's theorem.} \tag{1}$$

Also since (N) and (I) are tangent,

$$(x - c)^2 + y^2 = (R/2 - r)^2. \tag{2}$$

Eliminating r from (1) and (2), we get

$$4R^2[(x - c)^2 + y^2] = (x^2 + y^2)^2. \tag{3}$$

In a similar manner we can show that the ex-centers lie on the quartic (3). Transferring the origin to N and writing (3) in polar coördinates, we obtain

$$\rho^2 - 2c\rho \cos\theta - 2R\rho + c^2 = 0. \tag{4}$$

In the third term of (4) only the negative sign is used, as the positive sign gives imaginary values for ρ. For any fixed θ there are two values for ρ in equation (4), such that their product is c^2. The curve (4), therefore, consists of two ovals, each of which is the inverse of the other with respect to N as center and c as radius.

In equation (3) the substitution of $x - c = iy$ gives a perfect square, hence the point $(c, O) = N$ is a focus of (3).

DEFINITION: Let a radius vector of (4) cut the curve in T and T', and let S be the mid point of TT'.

THEOREM III: *A line l through S perpendicular to TT' envelops a circle with center H and radius R.*

PROOF: From equation (4) we find that $NS = c\cos\theta + R$.

Now draw through H a line parallel to NS cutting l in S', then $HS' = R$ for any value of θ. Therefore l is tangent to a circle with H as center and radius R.

THEOREM IV: *Tangents to the curve* (4) *at T and T' are equally inclined to TT'.*

THEOREM V: *The first polar of O with respect to* (3) *is a circle on NH as diameter.*

THEOREM VI: *The polars of any point on (O) with respect to (N) and (E) intersect on the line NH.*

The proofs of the above three theorems are simple and are therefore omitted.

THEOREM VII: *The locus of the orthopole of any chord of (O) is a circle equal to (N).*

PROOF: Let TT' be a chord of (O). Then the orthopole of TT' is the intersection S of the Simson lines of T and T'. Moreover the angle between the Simson lines of T and T' is constant and measured by one half the arc TT'. Also the Simson lines of T and T' pass through two fixed points P and P' on (N). Therefore the locus of S is a circle of which PP' is a chord subtending an angle equal to the angle which it subtends in (N). Hence the locus of S is a circle equal to the circle (N).

REMARKS: If TT' is a diameter, the locus of S is (N) itself.

In the pedal triangle DEF of ABC, H is the in-center, N is the circumcenter and the circle (N) is fixed. Hence DEF has a fixed inscribed circle and generates a poristic system of triangles. Also the antipedal triangle of O generates a poristic system of triangles.

THEOREM VIII: *The Nagel Point of ABC traverses an oval of a quartic similar to the quartic* (3).

Consider the antimedial triangle $A_1B_1C_1$. This has G for centroid, H for circumcenter and O for nine-point center and these points are fixed. Moreover the triangles ABC and $A_1B_1C_1$ are homothetic. Therefore the in-center of $A_1B_1C_1$ traverses an oval similar to the oval which I traverses. But the in-center of $A_1B_1C_1$ is the Nagel point of ABC. Hence the theorem.

Also since the circumcenter of the triangle $I_1I_2I_3$ is the symmetric of I with respect to O, this point also traverses an oval of a quartic similar to (3).

Let def and $d'e'f'$ be the pedal triangles of H and H', the inverse of H with respect to the circle (O). Then the triangles def and $d'e'f'$ are inversely similar.

THEOREM IX: *The locus of the double point of the inversely similar triangles def and $d'e'f'$ is the circle (N).*

Let T and T' be the ends of the diameter OH, and let ω be the orthopole of TT'. Now consider the figures $ATOT'$ and $\omega XA'X'$ (X and X' are the feet of the perpendiculars from T and T' on BC, A' is the midpoint of BC). $\angle TAT' = \angle X\omega X' =$ rt. $\angle$.

Also O and A' are the midpoints of TT' and XX' and $\angle OTA = \angle A'X\omega$. Therefore the figures $ATOT'$ and $\omega XA'X'$ are similar.

In these two figures H and d are homologous points. Also d' is homologous to H'. But AT bisects $\angle HAH'$, therefore ωX bisects $\angle d\omega d'$ and we have

$$\omega d : \omega d' = Xd : Xd' = TH : TH' \text{ etc.}$$

Therefore

$$\omega d : \omega e : \omega f = \omega d' : \omega e' : \omega f'.$$

Hence ω is the double point of the triangles def and $d'e'f'$. But from Theorem VII, ω is on (N).[1]

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

QUESTIONS.

The fourth of the discussions contained in this number suggests the following question:

52. Is there any simple treatment of the regular pentagon constructions the proof of which involves neither medial section nor trigonometric formulæ?

It is well known that simple constructions may be made based on the square root of five, and not unlike that given below; but works on formal geometry commonly follow Euclid's method.

DISCUSSIONS.

I. GEOMETRICAL CONSTRUCTION OF POINTS ON A FOUR-LEAF ROSE.

BY H. H. DOWNING, University of Kentucky.

Consider a fixed circle of radius a and a fixed diameter with extremities O and A. Through O draw a chord cutting the circle in R. Locate P on OR so that $OP = RM$, where M is the foot of the perpendicular from R on OA. As R describes the semicircumference AO, P will trace out one loop of the four-leaf rose $r = a \sin 2\theta$. It will be at once evident how to trace the other loops.

Proof: Taking OA as initial line with O as the origin, $OP = r$, the angle $AOR = \theta$, we have

$$r = OP = RM = OR \sin\theta = OA \cos\theta \sin\theta = 2a \cos\theta \sin\theta = a \sin 2\theta.$$

[1] In this connection see Gallatley, *Modern Geometry of the Triangle*, p. 47.

II. Note on the Solution of a Set of Linear Equations.

By J. P. Ballantine, Columbia University.

Let x_1, x_2, x_3, x_4 satisfy the set of equations

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= a_5, \\ b_1x_1 + \cdots &= b_5, \\ c_1x_1 + \cdots &= c_5, \\ d_1x_1 + \cdots &= d_5, \end{aligned}$$

supposed independent.

Then[1]

$$\begin{vmatrix} a_1x_1 - a_5 & a_2 & a_3 & a_4 \\ b_1x_1 - b_5 & b_2 & b_3 & b_4 \\ c_1x_1 - c_5 & c_2 & c_3 & c_4 \\ d_1x_1 - d_5 & d_2 & d_3 & d_4 \end{vmatrix} = 0;$$

for, on multiplying the respective columns by 1, x_2, x_3, and x_4, and adding, a column of zeros is obtained. From the vanishing of the above determinant may be obtained the usual formula for the value of x_1.

Suppose it is desired to solve the set of equations for only one of the unknowns, and to check the result. The usual method affords no check on the value of x_1 until all of the other unknowns have been evaluated; except, perhaps, the consistency of the four equations in the other three unknowns when the value of x_1 is substituted. This consistency condition is precisely the vanishing of the above determinant, a fact which is not usually brought out in an elementary treatment.

III. Evaluation of the Determinant $|1/(r+s-1)!|$.

By J. J. Nassau, Case School of Applied Science.

The evaluation of the determinant

$$D_n = \begin{vmatrix} \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{n!} \\ \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(n+1)!} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{(2n-1)!} \end{vmatrix}$$

has been given recently by L. L. Dines (*1923*, 196–8). In this note I propose to reduce the elements of D_n to binomial coefficients and by so doing to facilitate its evaluation.

[1] Dickson, *Elementary Theory of Equations*, 1914, p. 145.

Multiplying the first column by $n!$, the second by $(n+1)!$ and so on and dividing the first row by $(n-1)!$, the second by $(n-2)!$ and so on up to the $(n-1)$th row, which is divided by $1!$, we obtain

$$D_n = \frac{(n-1)!\,(n-2)!\cdots 1!}{n!\,(n+1)!\cdots(2n-1)!}\begin{vmatrix} \frac{n!}{1!\,(n-1)!} & \frac{(n+1)!}{2!\,(n-1)!} & \cdots & \frac{(2n-1)!}{n!\,(n-1)!} \\ \frac{n!}{2!\,(n-2)!} & \frac{(n+1)!}{3!\,(n-2)!} & \cdots & \frac{(2n-1)!}{(n+1)!\,(n-2)!} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & & 1 \end{vmatrix}.$$

By writing the nth row of this determinant as the first row, the $(n-1)$th row as the second and so on, and replacing the fraction $n!/(n-r)!\,r!$ by n_r, we have

$$(-1)^{n(n-1)/2}\begin{vmatrix} n_0 & (n+1)_0 & \cdots & (2n-1)_0 \\ n_1 & (n+1)_1 & \cdots & (2n-1)_1 \\ \cdots & \cdots & \cdots & \cdots \\ n_{n-1} & (n+1)_{n-1} & \cdots & (2n-1)_{n-1} \end{vmatrix}.$$

Determinants of this type have been considered by Zeipel [1] and lately by Muir.[2] The above special case can easily be shown to be equal to 1.[3]

Therefore

$$D_n = (-1)^{n(n-1)/2}\cdot\frac{(n-1)!\,(n-2)!\cdots 1!}{n!\,(n+1)!\cdots(2n-1)!}.$$

IV. On the Division of a Circumference into Five Equal Parts.

By H. C. Bradley, Massachusetts Institute of Technology.

The following simple construction for dividing a circumference into 5 or 10 parts is, in the opinion of the author, new and differs considerably from methods commonly described in text-books of plane geometry. Instead of constructing the sides of regular inscribed decagon and pentagon, this method constructs arc sec $(\sqrt{5}-1)$ and arc sec $(\sqrt{5}+1)$ and assumes that it has been proved by trigonometry that these are, respectively, 36° and 72°. The details follow:

At a point A on the circumference of a circle of radius R and center O, draw the tangent AH. On AH lay off $AC = 2R$. On OC, towards O, lay off $CD = R$. With center O and radius OD draw an arc intersecting AH at F. Then $\angle FOA = \text{arc sec}\,(\sqrt{5}-1) = 36°$.

On OC, away from O, lay off $CE = R$. With center O and radius OE draw an arc intersecting AH at G. Then $\angle GOA = \text{arc sec}\,(\sqrt{5}+1) = 72°$.

[1] Om Determinanter, hvars elementer ars Binomialkoefficienter. *Lunds Univ. Arsskrift*, ii (1865), pp. 1–68. (Muir's *Theory of Determinants*, Vol. III.)

[2] *Transactions of the Royal Society of South Africa*, 1923, Vol. XI, pp. 191–196.

[3] Muir's *Theory of Determinants*, Vol. III, p. 448.

V. On the Definition of Determinants.

By A. A. Bennett, University of Texas.

This brief article will treat of but one feature of the definition of determinants, namely that of the algebraic sign to be prefixed before any one of the given products of n elements which constitute the terms of the expansion of the determinant. This important question is that of the parity (oddness or evenness) of a permutation, and except for the significance of this particular application belongs in the theory of permutation groups. Regarded abstractly, a set of elements is given and a one-to-one reciprocal correspondence (called a permutation) of the set is also given, one asks whether the permutation is odd or even. Before this question can be answered in any given case it is essential that these terms be defined, and that we be convinced that each permutation must be either odd or even and can never be both.

Several methods of procedure are current but they usually agree by first establishing an arbitrary sequential order among the given elements, and even assigning labels to them as, for example, $e_1, e_2, e_3, \cdots, e_n$. A rearrangement of these is then tested for parity. This is ordinarily accomplished in one of two ways. One way is as follows. A set of n real distinct numbers, $x_1, x_2, x_3, \cdots, x_n$, is selected in one-to-one reciprocal correspondence with the e's, and hence with the given elements. The product, $X, = \Pi(x_i - x_j), i < j, i, j, = 1, 2, \cdots, n$, is constructed and it is noted that X can at most change sign if the quantities, x_i, be rearranged in any order. If X does not change sign under a given permutation, this permutation is said to be *even*, while if X does change sign, the permutation is *odd*. The other method is as follows. Let the elements, e_i, as rearranged by a permutation be denoted by $e_1', e_2', e_3', \cdots, e_n'$, where each e_i' is one of the original set, $(e_1, e_2, e_3, \cdots, e_n)$, under a different notation. A pair of distinct elements in the rearrangement, namely $(e_j', e_k'), j < k, j, k = 1, 2, \cdots, n$, defines a *permanence* if the subscript of e_j', when written as an e_i, is less than that of e_k' written as an e_i', otherwise the pair defines an *inversion*. In determining the number of permanences or of inversions, it is necessary to consider every pairing of distinct elements, and not merely pairs of successive elements. Thus the total set of permanences and of inversions in any given case will of course number exactly $n(n-1)/2$. The permutation is said to be *odd* or *even* according as the number of inversions is odd or even.

Both of these methods are open to the disadvantage that the definition involves a one-to-one reciprocal correspondence from the given set to the set of e's, $(e_1, e_2, e_3, \cdots, e_n)$. Until the contrary is established it is necessary to refer to a permutation as being odd or even *with respect to a given initial ordering*. The proof that the parity is independent of the ordering takes several steps and shows that the original definition introduced an extraneous item. The method of permanences and inversion might seem a trifle less artificial than the use of the algebraic function, X. On the other hand, the notion of permanence and that of inversion have only passing interest, and are merely incidental auxiliaries

to aid in the establishment of the concept of parity. The algebraic function, X, has further significance, being the square root of the discriminant, and plays a basic rôle in the theory of geometrical constructions, of alternating functions, of the algebraic solution of algebraic equations, and so forth. Despite these further uses of the same machinery, the objection stated still remains.

One might start with the real difficulty involved in the usual procedure, and define a permutation as odd or even according as it is the product of an odd or even number of transpositions. This leaves open the question as to whether the representation of a permutation as a product of transpositions is unique as to parity. It is nearly obvious that it is not unique as to the number or the sequence of the transpositions employed, so that one might reasonably question whether the parity is not dependent upon the particular analysis of the given permutation into transpositions.

It would seem desirable that texts should use a definition of parity not confused from the start by extraneous features or open to queries of this sort, even though such difficulties might be of short duration. If available, one would ask for a definition from which it is at once obvious that the parity is an inherent characteristic of the permutation. Whether or not the parity is related to the number of transpositions does not appear to be so essential in the first stages of the discussion. The parity having been once defined, the definition should be such as to make it readily seen that the product of any permutation by a transposition has a parity opposite to that of the given permutation. If one then proves that every permutation is the product in at least one way of a set of transpositions, the relation between parity and transpositions is completely explained. For then the identity, which is even, cannot be expressible as the result of an odd number of transpositions, so that if two distinct representations of a given permutation be at hand, each expressed as a continued product of transpositions, the product of the first representation by the inverse of the second would be the identity, and must involve an even number of transpositions, from which it is at once inferred that both of the given representations involved an odd number of transpositions or both involved an even number.

Fortunately one need not seek far for a characteristic property that might well serve as a definition, although, despite its familiarity, its advantages seem to have been generally ignored.

We shall assume that the notion of a permutation itself has been established. One may then study, as is the custom in some connections, the independent cycles generated in the set of elements by repetitions of the permutation. The facts that in a finite set the cycles may be listed in any order and that the elements in a cycle may be given any circular permutation without altering the defining permutation are almost self-evident and may be proved without mention of parity or the analysis of a permutation into the product of transpositions, except of course for such trivial examples as those in which no cycle is of greater order than 2. Furthermore there is no appeal to a notion of initial ordering. This is obvious from the fact that the cycles determine the permutation, and these

are subject to the rearrangements mentioned. The parity is now defined to be that of the total number of independent cycles each of which has an even number of elements. Since no extraneous correspondence is invoked, there is no occasion to prove that the parity is independent of a particular ordering. Incidentally the proof that a permutation is of like parity with its inverse becomes obvious without appeal to transpositions.

It seems surprising that so familiar a relation as that between the number of cycles of even order and the parity should be left so generally as a proposition to be proved upon the basis of an awkward definition of parity. This is perhaps due to one's tendency to regard any given finite set as having some initial sequential order as the basis of discussion, the fact that the order is immaterial being regarded as in no sense relieving us of thinking of the set as ordered. If, in teaching, one should use such symbols as "+", "−", "0", "*", or the names of several colors, or some other set for which no particular ordering at once suggests itself, there might be less temptation on the part of the student to think of the elements in a given discussion as inherently ordered. There is something more than merely mental discipline to be gained in thinking of sets as given without order, even in connection with the special applications of these notions to the initial problem of this article, namely, to the definition of determinants, which is the very topic in which so much energy is usually concentrated upon relations of order. Let A be a set of n symbols, and e_{ab}, where a and b are arbitrary elements of A, be a set of n^2 numbers. Consider a product of n symbols, e_{ab}, where the a's are distinct and the b's are also distinct. This product completely identifies a permutation in the set, A, as from the a's to the b's. This permutation depends upon the choice of the set of factors but is completely independent of the order in which they are written to form the product. If the permutation that is thus defined is odd, place a minus sign before this product, if even, place a plus sign. The algebraic sum of all such distinct signed products is defined as the determinant of the set of n^2 elements, e_{ab}. That an interchange of first and second subscripts throughout does not alter the determinant is proved with the greatest ease. In each term the new permutation is the inverse of the original one, and hence of the same parity. The new expression contains except for signs the terms necessary for the definition of a determinant, and we have just seen that the correct sign in the present case is the same as the sign of the terms from which each is secured. This may be stated as the proposition that a determinant is unaltered, if rows are changed into corresponding columns and columns into corresponding rows. In the case of several other familiar transformations which leave a determinant unaltered or at most change its sign, the proof is made appreciably simpler by using this definition of a determinant.

VI. Note on the Nature of the Correlation Coefficient.

By H. M. Roeser, Bureau of Standards.

The ideas brought out in Professor Dunham Jackson's paper "The Algebra of Correlation" in the Monthly (*1924*, 110) seem to give justification for the presentation of a brief note on the nature of the correlation coefficient which has not come to my attention in published form although my contact with literature on statistics has not been intimate during the past four or five years. While being of very elementary nature, it may be found useful in creating a working conception of the correlation coefficient in the minds of students of statistical theory.

As a starting point, consider section 9 of Professor Jackson's paper in which, after developing the equation for the line of regression, he further produces the familiar relation

$$\sigma_1^2 = \sigma^2(1 - r^2).$$

By solving for r the following relation is obtained,

$$r = \pm\sqrt{1 - \frac{\sigma_1^2}{\sigma^2}} = \pm\sqrt{1 - \frac{\Sigma v_k^2}{\Sigma y_k^2}}.$$

In this last expression the v's are what are termed "residuals" in the theory of least squares. In the correlation plot they are the vertical deviations of the y_k's from the line of regression. This being the case, a mental grasp of the correlation coefficient is immediately furnished especially if one has had previous experience with the method of least squares.

If the correlation coefficient is equal to unity, correlation is said to be perfect, that is, the values of y_k fall exactly upon the line of regression. This is directly apparent from the above equation because the coefficient cannot equal unity unless $\Sigma v_k^2 = 0$, which in turn requires that the values of y_k be exactly on the line.

If the correlation coefficient is zero, there is said to be no correlation, that is, the line of regression coincides with the x-axis. In this case each value of v is equal to the corresponding value of y_k, or each value of y_k is its own deviation from the line of regression. This is apparent in the above equation since it is the only condition under which the coefficient can be equal to zero.

It is further evident from the above equation that the correlation must vary from zero to unity in absolute value and the smaller the absolute value the poorer the agreement with the line of regression.

RECENT PUBLICATIONS.

EDITED BY D. C. GILLESPIE, Cornell University, to whom books and communications should be sent.

REVIEWS.

The Mathematics of Investment. By W. L. HART. New York, D. C. Heath and Co., 1924. 11 + 220 + 88 (tables) pages. Price $3.40 (without tables, $2.48).

The number of textbooks on finance is multiplying so rapidly just now that the first thing the reader of this review will probably want to see is a list of the features which are most apt to recommend this new book. The features which impress the reviewer the most are (1) the abundance of exercises; (2) the omission of much theory and work which have always been included in other textbooks on finance but which are of little value or importance; (3) good tables (to many decimal places and for many rates of interest); (4) "the segregation of optional methods and difficult topics" into *supplementary sections* which may, if desired, be omitted in a short or elementary course.

The number of exercises is almost too large. It is so large that the reviewer would rather not offer any judgment as to their general merit or arrangement.

The treatment of building and loan associations which has never deserved the prominence given it in other textbooks has been boiled down to two supplementary sections. The treatment of depreciation is restricted to a few essentials. The same is true of bonds. Even the treatment of probability is restricted to merely the fundamental definition; compound probability is not even mentioned. The laborious problems of determining the rate of interest to many decimal places in various situations based upon the use of annuities have been omitted but interpolation by proportion is given a rather prominent place.

Criticisms have often been made by individuals, usually unfamiliar with work in finance, that the universally adopted symbols are not sufficiently comprehensive or informing. The author of this book uses a modification which would probably prove quite acceptable to such critics. The reviewer, however, does not like the innovation. After years of experience in teaching, the reviewer has become convinced that the average student is strangely appalled not so much by any difficulties offered by a theory if concisely and clearly stated, as by the *appearances* of difficulty given by a printed page even though it involves no more than the four fundamental operations of arithmetic, and a printed page which seems to be very complicated will often give an impression which it is very difficult to dislodge. The reviewer believes that the author's innovation in symbols will produce just that effect. Moreover, the feature of the book which struck the reviewer as most undesirable is this general impression which many of the pages give—when the contents are really very simple. This effect is caused by both the notation just referred to and also the unfortunate jamming of too much work into a given page. It would have been much better if the work on each page had been given more space.

According to the preface only a most elementary knowledge of mathematics is needed for a proper study of the book but the reviewer feels that the introductions of such conceptions as limiting values (particularly that connected with the force of interest), convergency of series, the binomial theorem, etc., are not accompanied by explanations which will be clear to the student having only an elementary knowledge of mathematics.

On the whole, the book has been very carefully written and printed (no important errors were noted) and students who use the book properly are very apt to prove better drilled in the essentials of the mathematical theory of finance than those who use other current textbooks both because of the great variety and number of exercises and also because the treatment is restricted to essentials.

The book is divided into three parts. The first is entitled Annuities Certain; the second, Life Insurance; and the third, Auxiliary Subjects, including logarithms and progressions. Several topics are considered in an appendix. Answers to the exercises are issued in a separate booklet.

C. H. FORSYTH.

The Numerical Evaluation of the Incomplete B-*Function*, $\int_0^x x^{p-1}(1-x)^{q-1}dx$ *for Ranges of* x *between* 0 *and* 1. By H. E. SOPER. Cambridge University Press, 1921. 53 pages. Price 3s. 9d.

Table of the Logarithms of the Complete Γ-*Function (for Arguments* 2 *to* 1200, *i.e., beyond Legendre's Range*). By E. S. PEARSON. Cambridge University Press, 1922. 12 + 16 pages. Price 3s. 9d.

Log $\Gamma(x)$ *from* $x = 1$ *to* 50.9 *by Intervals of* .01. By JOHN BROWNLEE. Cambridge University Press, 1923. 23 pages. Price 3s. 9d.

These are respectively numbers VII, VIII and IX of Tracts for Computers edited by Karl Pearson and published by the Department of Applied Statistics of University of London, University College. We have already reviewed numbers I–IV (*1921*, 265–267) of these Tracts.

In Tract VII the following topics are considered: Integration in series attending index changes; Integration by polynomial approximations to the second factor, and other polynomial approximations; Trigonometric transformations and approximations; and Special cases of one index very large, both indices very large, and equal indices.

In many statistical problems this value of $\Gamma(p)$ is required for high values of p. It was to supply this need that the tables of Tract VIII for the new range of values of log $\Gamma(p)$, $p = 2$ to 1200, were published. From 2 to 5, p changes by tenths; from 5–10 by two-tenth intervals, and then by unit intervals to 1200. The table is to 10 places of decimals, and its use calls for a table of logarithms to 10 places such as the one by Duffield issued for the U. S. Coast and Geodetic Survey, Washington, D. C., in 1897.

Tract IX gives the values of log $\Gamma(p)$ to seven figures for values of p from 1 to 50.9 by intervals of .01 and thus supplements Tract VIII.

We have already given a bibliography of previously published tables of the gamma function (*1921*, 267).

R. C. ARCHIBALD.

Differential Equations in Applied Chemistry. By F. L. HITCHCOCK and C. S. ROBINSON. New York, John Wiley & Sons, 1923. vi + 110 pages. Price $1.50.

The purpose of the book, which is based on a recent course given to the chemical engineers at the Massachusetts Institute of Technology, is "not to teach Chemistry, but to teach Mathematics in a form readily assimilated by chemists and chemical engineers." More especially, the first object of this book is to help such students to think more readily "in terms of calculus." This, together with a general explanation of the fundamental concept of calculus and its value as a tool, is set out nicely in the preface and in a rather plastically written introductory chapter. As far as the reviewer can see, the latter is likely to attract chemists and to stimulate them to a neat and definite way of thinking and reasoning about problems.

The next two chapters are devoted to "processes of the first and of the second order," respectively, terms borrowed from the chemist's usual nomenclature. In plain mathematical language, these chapters treat the simple but all-important equations of the form

$$\frac{dx}{dt} = kx \tag{1}$$

and

$$\frac{dx}{dt} = k(a - x)(b - x), \tag{2}$$

whose integrals are described in a variety of ways and illustrated by over fifty practical examples.

In Chapter IV several types of simultaneous processes are treated, the corresponding mathematical problems being largely reducible to solutions of (1) and (2) and, in a few cases, to slightly more complicated equations, or rather evaluation of quadratures. The so-called "consecutive processes" offer a good opportunity for treating the linear equation

$$\frac{dx}{dt} + Px = Q,$$

where P, Q are functions of t. The subject is again illustrated by copious examples and exercises (problems, with hints).

Chapter V, equations of flow, acquaints the chemical reader with the one-dimensional case of Fourier's partial differential equation (heat conduction), *i.e.*,

$$\frac{\partial f}{\partial t} = a^2 \frac{\partial^2 f}{\partial x^2},$$

with a constant, and its integration by series. This chapter seems particularly promising with regard to its educative value.

The last chapter, VI, treats of the graphical evaluation of integral expressions, and is followed by an appendix containing forty-five miscellaneous problems.

The booklet, carefully printed and written in an agreeable style, can be warmly recommended not only to chemists, but to mathematical beginners of any other vocation.

L. SILBERSTEIN.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 30, no. 1–2, January–Febuary, 1924: "Note on stability à la Poisson" by F. H. Murray, 17–18; "Applicability with preservation of both curvatures" by W. C. Graustein, 19–23; "Complete sets of representations of two-element algebras" by B. A. Bernstein, 24–30; "Brouwer's contributions to the foundations of mathematics" by A. Dresden, 31–40; "A qualitative definition of the potential functions" by P. Franklin, 41–50; "On the location of the roots of polynomials" by J. L. Walsh, 51–62.

JOURNAL OF THE FRANKLIN INSTITUTE, volume 197, no. 4, April, 1924: "On stresses in a plate with a circular hole" by S. Timoshenko, 505–516.

JOURNAL OF MATHEMATICS AND PHYSICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, volume 3, no. 2, March, 1924: "In memory of Joseph Lipka" by N. Wiener, 63–65; "The coincident points of two algebraic transformations" by F. L. Hitchcock, 66–71; "The quadratic variation of a function and its Fourier coefficients" by N. Wiener, 72–94; "Note on operational calculus" by V. Bush, 95–107; "Note on quantization of non-conditioned-periodic systems" by M. S. Vallarta, 108–117; "A contribution to the generalization of a determinantal theorem of Frobenius" by L. H. Rice, 118–126.—No. 3, April, 1924: "The Dirichlet problem" by N. Wiener, 127–146; "On tautochronous motion" by S. D. Zelden, 147–161; "The theory of testimony" by J. S. Taylor, 162–173; "Notes on dynamical systems non-integrable by separation of variables and on the existence of 'unmechanical' orbits in the atom" by M. S. Vallarta, 174–181; "Note on a property of rectilinear lines of principal stress" by P. Heymans, 182–185; "Note on the curl" by W. H. Ingram, 186–187; "Note on a method of evaluating the complex roots of a quartic equation" by W. V. Lyon, 188–190.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 10, no. 4, April, 1924: "Electrodynamics of the general relativity theory" by G. Y. Rainich, 124–127; "Principal directions in an affine-connected manifold of two dimensions" by J. L. Synge, 127–129; "A statistical discussion of sets of precise astronomical measurements" by E. B. Wilson and W. J. Luyten, 129–132; "The quantum theory of the Fraunhofer diffraction" by P. S. Epstein and P. Ehrenfest, 133–139.—No. 5, May, 1924: "Prime power substitution groups whose conjugate cycles are commutative" by G. A. Miller, 166–167; "An extension of the theorem that no countable point set is perfect" by R. L. Moore, 168–170; "Concerning the prime parts of certain continua which separate the plane" by R. L. Moore, 170–175; "Concerning the division of the plane by continua" by J. R. Kline, 176–177.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 204, April, 1924: "Classroom devices in teaching algebra and geometry" by J. A. Nyberg, 345–349; "High fifth mathematics" by J. L. Green, 366–369; "Objections to mathematics" by A. J. Cave, 376–381.—May, 1924: "Where shall I place that decimal point?" by B. C. Zimmerman, 507–508; "Historical note on the solution of equations" by G. A. Miller, 509–510.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 25, no. 4, October, 1923: "Generalized limits in general analysis. Second paper" by C. N. Moore, 459–468; "The equilong transformations of Euclidean space" by B. H. Brown, 469–484; "Invariant sets of equa-

tions in Riemann space" by P. Franklin, 485–500; "Some properties of spherical curves, with applications to the gyroscope" by O. D. Kellogg, 501–524; "The greatest and the least variate under general laws of error" by E. L. Dodd, 525–539; "The intersection numbers" by O. Veblen, 540–550; "The geometry of paths" by O. Veblen and T. Y. Thomas, 551–608.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

PHI CHI MU OF WASHINGTON AND JEFFERSON COLLEGE, Washington, Pennsylvania.

In January 1920, Phi Chi Mu was established as an honorary fraternity to promote interest and stimulate activity in the study of mathematics, physics, chemistry and biology. The idea originated with two senior students majoring in mathematics and these met with the hearty coöperation of Professor C. S. Atchison. The heads of the departments of these subjects are honorary members and act as counsellors. Instructors are also elected as honorary members. The student membership is limited to twelve honor men majoring in mathematics, physics, chemistry or biology.

The officers for the year 1923–24 were H. L. Dorwart '24, president, and D. H. Rosenberg '24, secretary-treasurer. Most of the club members are majoring in mathematics, and consequently most of the papers presented are on mathematical subjects. Recent papers by undergraduate members were

1. "The philosophy of mathematics" by C. E. Lowery '24.
2. "The theory of parallels" by Boyd C. Patterson '23.
3. "The derivation and meaning of the Lorentz transformation in relation to Einstein's relativity" by H. L. Dorwart '24.
4. "The value of research" by D. H. Rosenberg '24.
5. "The life and work of Archimedes" by R. A. Klieves '24.

(Report by Professor C. S. Atchison.)

PI MU EPSILON OF THE UNIVERSITY OF ALABAMA.
[*1923*, 144.]

About a year and a half ago the Newtonian Club of the University of Alabama became a chapter of the Pi Mu Epsilon mathematical fraternity. The officers for the year 1923–24 are: President, George Shelton '24; secretary-treasurer, Daniel Clark '25; faculty director, Professor Tomlinson Fort. Meetings were held the last Tuesday in each month.

The programs were as follows:

October: Organization and business.

November: "Mathematics taught in the colleges of engineering," a statistical report with letters from many engineers, Professor Fort, Professor Dahlene, Professor Lewis, and George Shelton '24.

December: "Calculation of logarithms," Edward Duffy '25, "Slide rule," Professor Dahlene, "Trilinear coördinates," W. S. Ernst '24.

January: "Egyptian and Babylonian mathematics," Lucille Ellenburg '24, "Mathematics in literature," Corinne Alexander '25, "History of mathematics in America," Professor Fort.

February: Address by Professor H. E. Slaught of the University of Chicago.

March: "Cardan and Tartaglia," E. Dany '25, "Solution of the cubic," F. S. Gachet '24, "Trisection of the angle," O. A. Reed, '25, "Mathematics and music," C. E. Comeaux '25.

April: Solution of certain problems from the MONTHLY, F. S. Gachet '24, "The most pleasing rectangle," Esther Frank '25, "Graphical statics," George Shelton '24.

May: Business.

(Report by Professor Tomlinson Fort.)

THE MATHEMATICS CLUB OF COLUMBIA COLLEGE, New York City.
[*1922*, 77.]

The reorganization of the Mathematics Club of Columbia College, suggested by Walter F. Dantzscher '25, was intrusted by Professor Thomas Scott Fiske, executive officer of the department, to a student committee composed of Hector J. Battaglia '25, Walter F. Dantzscher '25, and Herbert A. Spurway '25. This committee with the approval of the membership continued to exercise its function as executive committee until the elections held in May. There were thirty students carried on the rolls as members. The average attendance was fifteen.

The programs were as follows:

December 17, 1923: Reorganization meeting. Introductory remarks concerning the reorganization of the club, Walter F. Dantzscher '25. "The Mathematics Club of Columbia College in the past," Albert E. Meder, Jr., assistant in mathematics and a former officer of the club. "The elementary theory of determinants," Herbert A. Spurway '25. Walter F. Dantzscher '25 presiding.

February 19, 1924: "Projective vector algebra," Hector J. Battaglia '25. Herbert A. Spurway '25 presiding.

March 4, 1924: "Numbers and the number concept," Walter F. Dantzscher '25. Hector J. Battaglia '25 presiding.

March 18, 1924: "Some curios of my collection," Professor D. E. Smith of Teachers College. Herbert A. Spurway '25 presiding.

April 29, 1924: "Some discontinuous functions," Professor Thomas Scott Fiske, executive officer. Walter F. Dantzscher '25 presiding.

May 13, 1924: Election of officers for the coming year. All elections were unanimous. President, Herbert A. Spurway '25; vice-president, Axel W. Berggren '25; secretary-treasurer, Joseph Ferony '26. A discussion of plans for the coming year also occupied a large part of the meeting, especially insofar as a constitution for the club was concerned.

Dr. G. A. Pfeiffer of the Department kindly acted as advisor to the Executive Committee.

(Report by Walter F. Dantzscher '25.)

THE MATHEMATICAL CLUB, NEW JERSEY COLLEGE FOR WOMEN, New Brunswick, N. J.

The following is the list of officers: President, Henrietta H. Dawson; vice-president and chairman of the program committee, Anne Dymock; secretary and treasurer, Dorothy McFarland.

During the year the following papers were read:

"Pythagoras, his life and works" by Miss Anne Dymock.
"Euclid, his life and works" by Miss Henrietta Dawson.
"Simson, his work in geometry" by Miss Margaret Dietrich.
"Measurement of time" by Professor W. E. Breazeale.
"Original theorems on Simson's line" by Mr. L. M. Paradiso.
"Linear perspective" by Professor A. A. Titsworth.
"The cyclic quadrilateral" by Professor R. Morris.
"Pseudo squares" by Miss Alna Ketterer.
"George Peacock, his life and works" by Miss Elizabeth Baier.
"De Moivre's theorem with applications" by Miss Ruth Thompson.
"Inscribed polygons" by Miss Anna Sebestyjak.

(Report by Professor Richard Morris.)

THE MATHEMATICS CLUB, RUTGERS COLLEGE, New Brunswick, N. J.

The following is the list of officers: President, N. Howard Ayers; vice-president, Robert M. Walter; secretary-treasurer, Wm. H. Mitchell, Jr.; faculty advisor, Professor Richard Morris.

During the year the following papers were read:

"Fixed points and circular loci," Professor Richard Morris.
"Simson's line and correlated theorems," Robert M. Walter.
Report on problems in the *Mathematics Teacher*, N. Howard Ayers.
Reports on various mathematical topics by members.
"Polygons and polygrams," Richard H. Cundy.
"De Moivre's theorem with applications," Addison Mallery.
"Application of trigonometric series," Simon Heimlich.
"Maxwell's law for the development of the distribution of the velocities of molecules in a gas," Charles J. Brasefield.

(Report by Professor Richard Morris.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3089. Proposed by NORMAN ANNING, University of Michigan.

Given four points, O, A, B, C, on a straight line, to construct, with straightedge only, the point P on the line such that OP shall be the harmonic mean of OA, OB, OC.

3090. Proposed by H. S. UHLER, Yale University.

Show that the volume of the (smallest) segment of a sphere (radius $= c$) cut out by two mutually perpendicular planes, the distances of which from the center are a and b respectively, may be expressed by the formula

$$\frac{2}{3}c^3\cos^{-1}\left[\left(\frac{a}{\sqrt{c^2-b^2}}\right)\left(\frac{b}{\sqrt{c^2-a^2}}\right)\right]-\frac{1}{3}b(3c^2-b^2)\cos^{-1}\left(\frac{a}{\sqrt{c^2-b^2}}\right)$$
$$-\frac{1}{3}a(3c^2-a^2)\cos^{-1}\left(\frac{b}{\sqrt{c^2-a^2}}\right)+\frac{2}{3}ab\sqrt{c^2-(a^2+b^2)}.$$

[*Note.* This formula may be used as major part of an alternative solution of problem **2947** [*1922*, 29] as given by J. B. Reynolds [*1923*, 209].]

3091. Proposed by PHILIP FITCH, Denver, Colorado.

The top of a grain hopper is in the form of a square whose side is 10 feet. The sides of the hopper are portions of right circular cylindrical surfaces, 20 feet in diameter. If the cylindrical surfaces meet the plane of the top at right angles and if the hole in the bottom of the hopper is one foot square, what is the volume of the hopper and how many bushels of grain will be taken out by lowering the level of the grain one foot?

3092. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

What must be the relation between the coefficients of a cubic equation in order that its roots, considered as lengths, shall form a triangle?

3093. Proposed by FRANK MORLEY, Johns Hopkins University.

Show that the equation

$$y = (a_0 + a_1x + \cdots + a_nx^n)/(b_0 + b_1x + \cdots + b_nx^n)$$

gives the differential equation

$$\begin{vmatrix} y_1, & y_2, & \cdots & y_{n+1} \\ y_2, & y_3, & \cdots & y_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n+1}, & y_{n+2}, & \cdots & y_{2n+1} \end{vmatrix} = 0,$$

where $y_r = (D_x{}^r y)/r!$.

3094. Proposed by A. A. BENNETT, University of Texas.

Given the set of numbers, a_{nr}, of double subscripts, $n = 1, 2, \cdots$, $r = 0, 1, 2, \cdots, n$, where $a_{10} = 1$, $a_{11} = 1$, $a_{20} = 1$, $a_{21} = 2$, $a_{22} = 2$, and in general thereafter $a_{n+1,\,r} = a_{nn} + a_{n,\,n-1} + \cdots + a_{n,\,n-r}$, where $a_{n,\,-1}$ may be regarded as equal to zero, and therefore $a_{n+1,\,0} = a_{nn} = a_{n,\,n-1}$. Show that a_{nn} is alternately odd and even. Note that for the first few even n's, a_{nn} is divisible

by a power of 2 of exponent comparable with n, *e.g.*, $a_{44} = 2^4$, $a_{66} = 2^4(1 + 2^4)$, $a_{88} = 2^8(1 + 2 + 2^2 + 2^3 + 2^4)$, $a_{10,10} = 2^{10}$ times an odd number, $a_{12,12} = 2^{12}(1 + 2^2 + 2^4 + 2^6 + \cdots + 2^{12})$, $a_{16,16} = 2^{16}$ times an odd number. Determine if possible an expression for a_{nn} in finite form, for at least even values of n, or failing in this, determine an asymptotic expression for its magnitude. In any case give $a_{24,24}$ explicitly. These numbers arise in connection with a counting of types of polynomials.

SOLUTIONS.

2959 [1922, 129]. Proposed by J. H. M. WEDDERBURN, Princeton University.
Solve the functional equation $[g(x)]^2 = -2x + g(x^2)$.

NOTE BY THE EDITORS.

See the paper by the proposer in the *Annals of Mathematics*, Dec., 1922, pp. 121–140, entitled "The functional equation $g(x^2) = 2\alpha x + [g(x)]^2$."

3030 [1923, 275]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.
Find the envelope of the bisector of the angle that a given segment subtends at a variable point of a given line.

NOTE BY THE EDITORS: See solution of **3035** which follows. Other solutions of **3030**, **3035** have been printed [1924, 312].

3035 [1924, 49]. Proposed by R. M. MATHEWS, Wesleyan University.
Generalize projectively and prove that the envelope of the bisectors of the angles between corresponding lines of two perspective pencils is a curve of the third class.

SOLUTION BY MABEL M. YOUNG, Wellesley College.

Let the corresponding rays of two perspective pencils P_1 and P_2 meet at Q on line q. Let the common ray be p. The external and internal bisectors of angles P_1QP_2 determine on p pairs of points of an involution of which P_1 and P_2 are double points. On the segments determined by these point-pairs as diameters construct circles. These form a coaxial system, S, and hence cut q also in pairs of points of an involution.

Let the bisectors of angle $P_1Q_1P_2$ meet p in B_1B_2. Since the bisectors are perpendicular, Q_1 lies on that circle of S which has diameter $\overline{B_1B_2}$. Let circle meet q again in Q_2. Since $\overline{Q_2B_1}$ and $\overline{Q_2B_2}$ are perpendicular and divide harmonically $\overline{Q_2P_1}$ and $\overline{Q_2P_2}$, they are bisectors of $P_1Q_2P_2$. Hence the bisectors of the angles with vertices at a pair of the involution on q pass by twos through a pair of the involution on p, and the four points lie on one circle of S. The involutions on p and q are then related in the following manner: To each of a pair of conjugate points on q corresponds one and the same pair of points in the involution on p. To each of a pair of conjugate points on p corresponds one and the same pair (real or imaginary) of the involution on q. Hence the two involutions are projective and corresponding pairs lie on the same circle of the system S. Since the circle on the point of intersection of p and q and its conjugate on p determines only one new point on q, this intersection is a self-corresponding point in the involutions.

Each point-pair in the involutions may be considered as a degenerate line conic. The two ranges in involution are then two projective pencils of line conics. The four lines joining the points of two corresponding pairs in the involutions become the common lines of corresponding conics of the pencils. By the theory [1] of the projective generation of curves these lines envelope a curve of class four with two double lines. Since however the intersection of the ranges is a self-corresponding point in the involutions, the four tangents which can be drawn to the curve from any point of the plane include three proper tangents, and a line through the self-corresponding point. The curve accordingly breaks into this point and a curve of class three to which the bases of the ranges are simple tangents.

We may then conclude that the external and internal bisectors of angles formed by corresponding rays of two perspective pencils determine on the axis of perspective and on the common ray corresponding point-pairs of two projective involutions with a self-corresponding element and hence envelope a curve of the third class.

3037 [1923, 337]. Proposed by E. O. BROWER, Chicago, Illinois.
The angles A, B, C of a triangle are given. A logarithmic spiral is tangent to AB at B and to AC at C; at what angle does the curve cut each radius vector?

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[1] Fiedler, *Darstellende Geometrie*, vol. III, B, pp. 37–45.

SOLUTION BY C. K. ROBBINS, Purdue University.

The equation of the logarithmic spiral is $r = ke^{a\theta}$. If ψ is the angle between the radius vector for any point and the tangent at that point, then $\tan \psi = rd\theta/dr = 1/a$. Let the inclinations of the tangents at the two points $B(r_1, \theta_1)$ and $C(r_2, \theta_2)$ be φ_1 and φ_2, respectively; and suppose that the tangents intersect in A. Denote the angles of the triangle ABC by A, B, C.

Consider first the case in which $\theta_2 = \beta_2$, $\theta_1 = 2\pi + \beta_1$, where β_1 and β_2 are positive angles less than π. For simplicity suppose $\beta_1 < \pi/2$ and $\beta_2 > \pi/2$. We shall then have

$$\theta_1 - \theta_2 = 2\pi + \beta_1 - \beta_2, \qquad \beta_1 = \varphi_1 - \psi, \qquad \beta_2 = \varphi_2 - \psi + \pi, \qquad \varphi_1 - \varphi_2 = A.$$

Hence

$$\theta_1 - \theta_2 = \pi + A.$$

If the angles OBC, OCB, O being the pole, be denoted by α_1, α_2, respectively, then

$$\alpha_1 = \pi - \psi - B, \qquad \alpha_2 = \psi - C.$$

It now follows that

$$e^{a(\theta_1 - \theta_2)} = \frac{r_1}{r_2} = \frac{\sin \alpha_2}{\sin \alpha_1}.$$

Inserting the above values, expanding the right-hand side and simplifying the result, we obtain

$$e^{a(\pi + A)} = \frac{\cos C - a \sin C}{\cos B + a \sin B}.$$

The value of a may be found by solving this equation by the usual approximation process when the values of A, B, C are given. An examination of the general case would show that multiples of π should be introduced in several of the above equations for the angles and also a number of $\pm$ signs should be placed before several of the angles. The resulting equation would be of the same type as the above with $\pm$ signs before the angles and with π replaced by $n\pi$ where n is an integer.

Also solved by W. B. CARVER, WILLIAM HOOVER, and J. B. REYNOLDS.

3041 [1923, 402]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two fixed intersecting planes on a variable line passing through a fixed point.

I. SOLUTION BY GRACE M. BAREIS, Ohio State University.

(*a*) Let the two fixed planes and any plane through the fixed point U be chosen as the reference planes $x = 0$, $z = 0$, $y = 0$ for a system of oblique Cartesian coördinates. Let the coördinates of U be $(a, 0, c)$. Any line through U is

$$\frac{x - a}{l} = \frac{y}{m} = \frac{z - c}{n},$$

where l, m, n are the direction ratios of the line. This line intersects $x = 0$ in a point

$$S\left[0, -\frac{am}{l}, -\frac{an}{l} + c\right]$$

and it intersects $z = 0$ in a point

$$T\left[-\frac{cl}{n} + a, -\frac{cm}{n}, 0\right].$$

Let $P(x, y, z)$ be the mid-point of ST; then

$$2x = a - \frac{cl}{n}, \qquad 2y = -m\left(\frac{a}{l} + \frac{c}{n}\right), \qquad 2z = c - \frac{an}{l}.$$

Eliminating l, m, n, we have as the locus of P $2xz = cx + az$, a hyperbolic cylinder through U, having the intersection of the fixed planes as an element, and having $z = \frac{1}{2}c$ and $x = \frac{1}{2}a$ as asymptonic planes.

(*b*) This problem is a special case of the following more general theorem: *Given a quadric surface S, a fixed plane ρ, and a fixed point U that is not the pole of ρ with respect to S; then any line a through U meets ρ in a point A whose polar plane α passes through R the pole of ρ.* α meets a in a

point P conjugate to A with respect to S. Now it is evident that $U(a, b, c, \cdots) \overline{\wedge} \rho(A, B, C, \cdots) \overline{\wedge} R(\alpha, \beta, \gamma \cdots)$. Therefore, $U(a, b, c, \cdots) \overline{\wedge} R(\alpha, \beta, \gamma, \cdots)$ and therefore all points such as $a\alpha$, $b\beta$, $c\gamma$, $\cdots$ lie on a quadric surface, *i.e.*, the locus of P is a quadric surface. This locus passes through U and R, through the points common to ρ and S, and through the points of contact of tangents from U to S if such points exist.

If ρ is the plane at infinity, then P is the conjugate of an infinitely distant point and is therefore the mid-point of the chord through U, R is the center of S, and we have the theorem: *The locus of the mid-points of all chords of a quadric surface that pass through a fixed point is another quadric surface through the fixed point and the center of the given quadric.*

Again if ρ is the plane at infinity and S is a degenerate quadric, a pair of intersecting planes, we have the theorem: *The locus of the mid-point of the segment determined by two fixed intersecting planes on a variable line through a fixed point is a quadric surface through the fixed point and the line of intersection of the two planes, this line being the line of centers of the degenerate quadric S.* Moreover, this quadric passes through the points common to ρ and S, that is, through the lines at infinity in the two planes. The locus is a hyperbolic cylinder having its asymptotic planes parallel to the given planes. [v. Reye, *Geometrie der Lage* (1882), vol. I, p. 101, and vol. II, p. 35.]

II. Solution by Mabel M. Young, Wellesley College.

Let a pencil of planes π be passed through that line p on the fixed point P which is parallel to the intersection l of the fixed planes. Each plane π cuts the given planes in two lines parallel to l. A third parallel x midway between these two lines is the locus of the mid-points of the segments determined by the fixed planes on all the lines through P which lie in π. The infinitely distant line in π is the harmonic conjugate of x as to the intersections of π with the fixed planes. Hence the planes $\lambda(l, x)$ and $\pi(p, x)$ meet the infinitely distant plane in lines which are harmonic conjugates in an involution of parallel lines, and the two pencils of planes are accordingly projective. Since the axes of these projective pencils of planes are parallel, the required locus is a cylinder.

The section of the fixed planes by any other plane on P gives a triangle bounded by two fixed lines and the infinitely distant line. The mid-point of the segment determined on any line through P by the sides of the triangle in the finite part of the plane is the fourth harmonic of the infinitely distant point in which it meets the third side. By a known theorem, the locus of the fourth harmonic of the three points in which the sides of a triangle are met by a variable line on a point in its plane is a conic through the vertices of the triangle and the fixed point. In this case the conic has two distinct points at infinity. Hence, the section of the required locus by any plane on P not parallel to the intersection of the fixed planes is an hyperbola through P. The locus is, accordingly, a hyperbolic cylinder through the intersection of the fixed planes and the fixed point. The given planes determine the direction of the asymptotic planes of the cylinder.

Also solved by William Hoover, H. Halperin, R. A. Johnson, A. Pelletier, J. B. Reynolds, and C. K. Robbins.

Note by the Editors: See **2904** [*1923*, 338] for the problem in the plane corresponding to the case of ρ as the plane at infinity. There it is shown that the locus is similar to the original curve.

3042 [1923, 402]. Proposed by C. N. Schmall, New York City.

Given the bases and the sum of the areas of several triangles that have a common vertex; show that the locus of the vertex is a straight line.

Solution by L. R. Ford, The Rice Institute.

We shall interpret the problem to mean that all areas are to be considered positive and that no deduction is to be made for overlapping areas.

Let n be the number of triangles. The n straight lines on which the bases lie divide the plane into regions, not exceeding $\frac{1}{2}(n^2 + n + 2)$ in number, and the locus is different according as the common vertex lies in one region or another. Designate the regions by R_1, R_2, etc.

Let $d_1, d_2, \cdots, d_n$ be the lengths of the bases; and let

$$a_i x + b_i y + c_i = 0 \qquad (1)$$

be the line on which the ith base lies. We shall suppose that this equation is in Hesse's normal

form, and that for a point (x, y) in one of the regions, R_1 say, the first member is positive. Then the first member is equal to the perpendicular distance from (x, y) to the line.

Let $P(x, y)$ be a point on the locus in R_1. Then the area of the ith triangle is

$$L_i(x, y) = \tfrac{1}{2}d_i(a_ix + b_iy + c_i), \tag{2}$$

and the equation of the locus is

$$L_1(x, y) + L_2(x, y) + \cdots + L_n(x, y) = A, \tag{3}$$

where A is the given constant area. Since this is linear in x and y, the locus is a straight line; and that portion of the line, if any, lying in R_1 is a part of the locus.

Pass now into an adjacent region R_2, by crossing the line of, say, the first base. In R_2 $L_1(x, y)$ is negative, and to keep the area positive we must change its sign. We have then another line

$$-L_1(x, y) + L_2(x, y) + \cdots + L_n(x, y) = A. \tag{4}$$

Since $L_1(x, y) = 0$ is a consequence of (3) and (4), it follows that these two lines meet on the common boundary of the two regions.

The locus for any region R_j is easily written down by merely changing the signs of certain terms in (3). If R_1 and R_j are on opposite sides of the line $L_i(x, y) = 0$, then the sign preceding $L_i(x, y)$ in (3) is changed from plus to minus. We get the equation of a straight line, and that part, if any, lying in R_j belongs to the locus.

At the boundary where a segment of the locus lying in one region terminates the segment lying in the adjacent region begins, so that we get for the locus a broken-line figure. We shall prove that *with the exceptions to be presently noted, these line segments form a closed convex polygon.*

The first exception arises in connection with an infinite region, where conceivably the locus may extend to infinity. However, as P moves along the line to infinity the area of each triangle increases without limit unless the locus is parallel to the base. Hence, this case can arise only if the bases lie on parallel lines.

A second exception occurs when the equation of the locus reduces to an identity; then all points of the region belong to the locus. The simplest example of this is the case in which the bases form the sides of a convex polygon and A is equal to the area of the polygon.

Putting aside these special cases we see that the locus consists of one or more closed broken-line figures. (The closure follows from the fact that each line terminates in others at its ends and that their number is finite.) That the locus consists of a single convex polygon will follow from the following proposition: *No three points on the locus which lie in different regions lie on a line.*

Suppose the contrary. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be collinear, P_3 lying on the segment P_1P_2. Choose the signs of the L's so that at P_3 all the terms of (3) are positive. Let $M_1(x, y)$ be the sum of the terms which are positive also at P_1 and P_2; let $M_2(x, y)$ be the sum of those which are positive at P_1 and negative at P_2; and let $M_3(x, y)$ be the sum of those which are positive at P_2 and negative at P_1. There are no terms negative at both P_1 and P_2, since no term can change sign twice as a point (x, y) moves along the line from P_1 to P_2. Substituting the coördinates of the points in the respective equations of the locus, we have

$$\begin{cases} M_1(x_1, y_1) + M_2(x_1, y_1) - M_3(x_1, y_1) = A, \\ M_1(x_2, y_2) - M_2(x_2, y_2) + M_3(x_2, y_2) = A, \\ M_1(x_3, y_3) + M_2(x_3, y_3) + M_3(x_3, y_3) = A, \end{cases} \tag{5}$$

where all the M's are positive. Now x_3 and y_3 can be written in the form

$$x_3 = lx_1 + mx_2, \qquad y_3 = ly_1 + my_2,$$

where l and m are positive and $l + m = 1$. Since each of the M's is linear we have

$$\begin{aligned} M(x_3, y_3) &= \alpha x_3 + \beta y_3 + \gamma \\ &= \alpha(lx_1 + mx_2) + \beta(ly_1 + my_2) + \gamma \\ &= lM(x_1, y_1) + mM(x_2, y_2). \end{aligned}$$

The third equation of (5) can then be written

$$l[M_1(x_1, y_1) + M_2(x_1, y_1) + M_3(x_1, y_1)] + m[M_1(x_2, y_2) + M_2(x_2, y_2) + M_3(x_2, y_2)] = A.$$

Multiplying the first equation of (5) by l and the second by m and subtracting both from the third, we have

$$2lM_3(x_1, y_1) + 2mM_2(x_2, y_2) = 0.$$

Since the terms in the first member are negative this equation is impossible,[1] and the proposition is established.

Two further facts may be noted. Parts of the locus lying in two regions on opposite sides of all the bases are parallel. For, the terms in the first members of the two equations all differ in sign and the coefficients of x and y in one case are therefore proportional to those in the other. By similar reasoning, the parts of the loci in a given region for various values of A are parallel.

The problem can be interpreted in mechanical terms. If each base represent a vector force so directed that its moment about a point in a given region is positive, then it is required that the sum of the moments of the forces about P be $2A$. Replacing the several forces by their resultant, we see that P must lie on a line parallel to the line of action of the resultant. The exceptional case in which the locus is an area will occur when the forces are equivalent to a couple, provided A has a suitable value.

Solutions of the problem in the sense intended by the PROPOSER were received from H. HALPERIN, WILLIAM HOOVER, R. A. JOHNSON, S. A. LUBKIN, A. PELLETIER and C. K. ROBBINS.

3043 [1923, 403]. Proposed by O. D. KELLOGG, Harvard University.

Let T denote an open continuum of the xy-plane, say the interior of a smooth simple closed curve. Then if U is continuous in T, and is such that

$$\iint_T U\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)dxdy = 0 \tag{1}$$

for all functions V with continuous second derivatives and vanishing on the boundary of T, U is harmonic, *i.e.*, satisfies Laplace's equation $(\partial^2 U/\partial x^2)+(\partial^2 U/\partial y^2)=0$. (It is understood that if U becomes infinite in the neighborhood of a boundary point of T, V is so further restricted that the integral shall have a sense.)

SOLUTION BY H. E. BRAY, Rice Institute.

Let P be any interior point of T. With center at P draw a small circle of radius r, C_r, entirely interior to T.

Let φ be a function defined as follows: $\varphi = 1$ inside C_r, $\varphi = 0$ otherwise.

For our function V, we shall take the approximating (average) function:

$$\varphi^{\nu\nu\nu}(x, y) = \frac{1}{(\pi\nu^2)^3}\iint_{C_\nu} d\xi''d\eta'' \iint_{C_\nu} d\xi'd\eta' \iint_{C_\nu} \varphi(x+\xi+\xi'+\xi'', y+\eta+\eta'+\eta'')d\xi d\eta,$$

where C_ν represents the interior of the circle $\xi^2+\eta^2=\nu^2$.

φ being integrable and bounded, it follows that $\varphi^{\nu\nu\nu}$ is continuous with its second derivatives. For the function

$$\varphi^{\nu}(xy) = \frac{1}{\pi\nu^2}\iint_{C_\nu} \varphi(x+\xi, y+\eta)d\xi d\eta$$

is continuous and

$$\frac{\partial\varphi^{\nu\nu}}{\partial x} = \frac{1}{\pi\nu^2}\int_{C_\nu}\varphi^{\nu}dy, \qquad \frac{\partial\varphi^{\nu\nu\nu}}{\partial x} = \frac{1}{\pi\nu^2}\int_{C_\nu}\varphi^{\nu\nu}dy.$$

The continuity of $\partial\varphi^{\nu\nu}/\partial x$ follows from the first of these equations and that of $\partial^2\varphi^{\nu\nu\nu}/\partial x^2$ from the last.

Moreover $\varphi^{\nu\nu\nu}$ is identically zero outside of a circle with center at P and radius equal to $r+3\nu$; and $\lim_{\nu=0}\varphi^{\nu\nu\nu}=\varphi$ inside and outside C_r; on C_r $\lim \varphi^{\nu\nu\nu}=\frac{1}{2}$. Thus $\varphi^{\nu\nu\nu}$ satisfies the conditions required of V provided ν is sufficiently small.

[1] $M_3(x_1, y_1)$ and $M_2(x_2, y_2)$ are zero if and only if P_1 and P_2 lie on the boundary of the region in which P_3 lies and at the termini of the segment of the locus through P_3. The three points could be considered as lying in the same region; so obviously this case should be ruled out.

Instead of U consider the approximating function $U^{\mu\mu}$.

$$U^{\mu\mu}(xy) = \frac{1}{(\pi\mu^2)^2}\int\int_{C_\mu} d\xi' d\eta' \int\int_{C_\mu} U(x+\xi+\xi', y+\eta+\eta')d\xi d\eta.$$

Thus $U^{\mu\mu}$ is continuous with its second derivatives. Now

$$\int\int_T U^{\mu\mu}\nabla^2\varphi^{\nu\nu\nu}dxdy = \frac{1}{(\pi\mu^2)^2}\int\int_{C_\mu} d\xi' d\eta' \int\int_{C_\mu} d\xi d\eta \int\int_T U(x+\xi+\xi', y+\eta+\eta')\nabla^2\varphi^{\nu\nu\nu}dxdy,$$

and since $\nabla^2\varphi^{\nu\nu\nu}$ vanishes identically outside the circle with center P and radius $r + 3\nu$ it follows, from our hypothesis, that if μ is sufficiently small,

$$\int\int_T U(x+\xi+\xi', y+\eta+\eta')\nabla^2\varphi^{\nu\nu\nu}dxdy = 0,$$

for all (ξ, η), (ξ', η') such that $\xi^2 + \eta^2 \leq \mu^2$, $\xi'^2 + \eta'^2 \leq \mu^2$.
Hence,

$$\int\int_T U^{\mu\mu}\nabla^2\varphi^{\nu\nu\nu}dxdy = 0.$$

We can now apply Green's Theorem to $U^{\mu\mu}$ and $\varphi^{\nu\nu\nu}$. Since $\varphi^{\nu\nu\nu}$ vanishes with all its derivatives everywhere outside the circle with center at P and radius equal to $r + 3\nu$, we obtain

$$\int\int_T \nabla^2 U^{\mu\mu}\varphi^{\nu\nu\nu}dxdy = 0.$$

Now let ν approach zero; $\varphi^{\nu\nu\nu}$ approaches φ and

$$\lim \int\int_T \nabla^2 U^{\mu\mu}\varphi^{\nu\nu\nu}dxdy = \int\int_{C_r} \nabla^2 U^{\mu\mu}dxdy = 0,$$

and since C_r is a circle of arbitrarily small radius

$$\nabla^2 U^{\mu\mu} = 0$$

at P, *i.e.*, $U^{\mu\mu}$ is harmonic at every interior point of T.

Apply the average value theorem to $U^{\mu\mu}$. For any circle C_a of small radius about any point P of T as center

$$U^{\mu\mu}(P) = \frac{1}{2\pi a}\int_{C_a} U^{\mu\mu}ds.$$

Let μ approach zero. Since U is continuous $U^{\mu\mu}$ approaches U uniformly on C_a. Hence

$$U(P) = \frac{1}{2\pi a}\int_{C_a} Uds,$$

i.e., the average value of U on the circumference of any sufficiently small circle inside T is equal to the value at the center. Hence by the definition of $U^{\mu\mu}$ we see that $U^{\mu\mu}$ and U are equal if 2μ is less than the distance of the point P from the boundary of T. U is therefore harmonic.

N. B. It can be proved similarly, by considering the function $U^{\mu\mu\mu}$ instead of $U^{\mu\mu}$, that the theorem remains true, with a certain reservation, if we assume that U is summable in the Lebesgue sense without being continuous. In fact U may then have removable discontinuities; but when these are removed, by redefinition at points of a set of measure zero, U becomes harmonic.

Also solved by NORBERT WIENER and the PROPOSER. Their solutions are different from the above, but agree with each other in using Koebe's converse of Gauss' theorem, to the effect that if U is the arithmetic mean, at each point, P, of T, of its values on the circumference of every circle with center at P and lying in T, then U is harmonic. The desired result is then obtained by what amounts to the use of the DuBois-Reymond form of the fundamental lemma of the Calculus of Variations.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, 22 East 38th St., New York City.

Miss KATHERINE S. ARNOLD, for the past three years professor of mathematics at Constantinople Woman's College, has been appointed a member of the administrative staff of the American Association of University Women at the National Headquarters, Washington, D. C. Miss Arnold will assume her new duties in September, 1924.

Adjunct Professor P. M. BATCHELDER, of the University of Texas, has been appointed acting assistant professor of mathematics at Brown University for the academic year 1924–1925.

Mr. D. H. MACPHERSON, of Brown University, has been appointed instructor of mathematics at the Brooklyn Polytechnic Institute.

At Cornell University, Assistant Professor W. A. HURWITZ has been promoted to a full professorship of mathematics. Professor C. F. CRAIG was on leave of absence for the second semester of 1923–1924, and Professor D. C. GILLESPIE has been granted leave of absence for the academic year 1924–1925.

Dr. D. S. MORSE, of Cornell University, has been appointed assistant professor of mathematics at Union College.

Assistant Professor W. L. G. WILLIAMS, of Cornell University, has been appointed assistant professor of mathematics at McGill University.

At Wells College, Professor T. R. HOLLCROFT has been granted leave of absence for the first semester of 1924–1925, and will go to Italy to study at the University of Rome. Miss EVELYN T. CARROLL has been promoted to an assistant professorship and will be acting head of the department during the absence of Professor Hollcroft. Miss FRANCES THOMAS has been appointed instructor of mathematics for the year 1924–1925.

Dr. R. F. BORDEN, of Brown University, has been appointed assistant professor of mathematics at George Washington University, Washington, D. C.

Miss NELLIE C. STOKES, of Brown University, has been appointed instructor of mathematics and English at Coker College, Hartsville, S. C.

Miss LESLIE GAYLORD, of Agnes Scott College, Decatur, Georgia, has been promoted to an assistant professorship of mathematics.

At Shorter College, Rome, Georgia, Professor RUBY U. HIGHTOWER has been granted leave of absence for the academic year 1924–1925, which she will spend in study at the University of Missouri. Miss EDNA ROBINSON, of the University of Missouri, has been appointed to fill the vacancy for the year.

Miss JULIA L. HAWKINS, associate professor of mathematics at the Oklahoma College for Women, has been appointed Dean of Women.

Professor J. M. HOWIE of Alma College, formerly of the Peru State Normal School, will teach in the summer session at Greeley, Colorado, and in September will assume the headship of the department of mathematics at Nebraska Wesleyan University. He has spent the past year in graduate work at Columbia University.

Mr. E. P. MARTINSON, a graduate of the University of Nebraska, has been appointed instructor in mathematics in the Colorado School of Mines.

Mr. A. E. ANDERSON, a graduate of the University of Nebraska, has been appointed instructor in mathematics in the University of Oklahoma.

The following 28 doctorates with mathematics as major subject were conferred by American universities in the calendar year 1923; the university and the title of the dissertation are given with each name. E. F. Allen, Missouri, "A revision of certain types of the Lie theory;" Constance R. Ballantine, Chicago, "Modular invariants of a binary group with composite modulus;" J. P. Ballantine, Chicago, "A postulational introduction to the four color problem;" A. D. Campbell, Cornell, "The classification of linear systems of conics in various domains;" W. E. Cleland, Princeton, "Permutable transformations F and transformations W;" M. M. Feldstein, Chicago, "Invariants of the linear group, modulo p^r;" C. A. Garabedian, Harvard, "A method of series in elasticity, with applications (1) to circular plates of constant or variable thickness, and (2) to rods of constant or variable circular cross-section;" R. E. Gleason, Princeton, "On a calculus of average value functions;" B. Z. Linfield, Harvard, "On the theory of discrete variates;" N. B. MacLean, Chicago, "On certain surfaces related covariantly to a given ruled surface;" D. S. Morse, Cornell, "Relative inclusiveness of certain definitions of summability;" F. H. Murray, Harvard, "The real solutions of certain systems of differential equations;" H. L. Olson, Chicago, "Congruences with constant absolute invariants;" J. O. Osborn, Cornell, "A study of the rational involutorial transformations in space which leave a web of sextic surfaces invariant;" G. E. Raynor, Princeton, "Dirichlet's Problem;" Emeterio Roa, Michigan, "A number of new generating functions, with applications to statistics;" J. B. Scarborough, Johns Hopkins, "The aerodynamics of a warped airplane wing;" G. E. F. Sherwood, Chicago, "Equivalence of triples of bilinear forms;" C. A. Shook, Johns Hopkins, "The distribution of lift over thin wing sections;" J. M. Thomas, Pennsylvania, "Congruences of circles studied with reference to the surface of centers;" T. Y. Thomas, Princeton, "Geometries of paths admitting first integrals;" C. E. Van Horn, Chicago, "A system of relative existential propositions connected with the relation of class membership;" F. M. Weida, Iowa, "The valuation of life annuities with refund of an arbitrarily assigned part of the purchase price;" Louis Weisner, Columbia, "Groups whose maximum cyclic subgroups are independent;" Miss A. M. Whelan, Johns Hopkins, "The theory of the binary octavic;" R. L. Wilder, Texas, "Concerning continuous curves;" R. E. Wilson, Chicago, "Representations of certain functions of two variables by Stieltjes integrals;" B. F. Yanney, Chicago, "Modular invariants of the binary quartic."

As is known to MONTHLY readers, the NATIONAL RESEARCH COUNCIL is now offering research fellowships in mathematics as well as in physics and chemistry. These fellowships are intended for men of the greatest promise in mathematical research. The Fellowship Board meets four times a year, so that it is possible to make application at any time during the year. The mathematical representatives of the Board which awards the fellowships are Professor G. A. BLISS of the University of Chicago and Professor OSWALD VEBLEN of Princeton University. Applications should be sent directly to Professor W. E. TISDALE, Executive Secretary, Research Fellowship Board, Washington, D. C.

This Board met on April 24, and the following appointments in Mathematics were made: L. M. GRAVES, Ph.D., June, 1924, Chicago; H. LEVY, Ph.D., June, 1924, Princeton; J. H. TAYLOR, Ph.D., June, 1924, Chicago; J. M. THOMAS, Ph.D., 1923, Pennsylvania. Also the following re-appointments in Mathematical Physics: H. ZANSTRA, Ph.D., 1923, Minnesota; T. Y. THOMAS, Ph.D., 1923, Princeton.

Attention is again called to the coöperative relationship between the Association and the *Annals of Mathematics*. This arrangement enables members of the Association to subscribe for the *Annals* at one half the regular subscription rate, namely, $1.50 instead of $3.00. It has long been the hope of the editors of the MONTHLY that financial conditions might eventually warrant the enlargement of the MONTHLY volumes so as to give space for more expository papers. In lieu of this desirable condition, the Association has for several years given a subsidy to the *Annals* in return for which the *Annals* has published numerous and extensive expository papers and has also given to Association members the special half-price subscription privilege. It is soon to publish an eighty page exposition of the Lie Theory of Differential Equations by Professor L. E. DICKSON. Reprints of such expository papers are for sale by the *Annals*. The cost for this particular one will doubtless be well toward that of a subscription for an entire year at the special rate to Association members. Subscriptions should be sent directly to the *Annals of Mathematics*, Princeton, N. J., the volume beginning with the issue for Septembes of each year.

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THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

WALTER BURTON FORD, EDITOR-IN-CHIEF
HERBERT ELLSWORTH SLAUGHT
AUBREY JOHN KEMPNER

WITH THE COÖPERATION OF

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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME XXXI, 1924
NUMBER 8, OCTOBER

PUBLISHED BY THE ASSOCIATION
LANCASTER, PA., AND ANN ARBOR, MICH.

Entered at the Post Office at Lancaster, Pa., as Second Class Matter
PUBLISHED MONTHLY

$4.00 a Year, Single Copies 45 cents, to Members;
$5.00 a Year, Single Copies 60 cents, to Others